Congestion with heterogeneous commuters

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A B S T R A C T

We study a congestion model where a continuum of heterogeneous commuters make a binary choice between riding a bus and driving private vehicles for their commutes. Formulating the model as a large game, we establish the existence and uniqueness of a nontrivial Nash equilibrium and analyze how a gasoline tax affects the allocation of commuters between public transportation and private vehicles at the equilibrium. Based on the analysis, we provide a sufficient condition under which a gasoline tax is Pareto improving. We also prove the existence of a socially optimal policy that minimizes the aggregate loss to all commuters.

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1. Introduction

Most American cities have suffered from traffic congestion for decades; this problem is likely to worsen with increasing urbanization and growth of population. For individual commuters in the United States, the average annual peak hours delay increased from 14 h in 1982 to 38 h in 2005; for the nation as a whole, the hours of delay rose from 0.8 to 4.2 billion for the same time period. The cost associated with the traffic congestion (in 2005 dollars) is substantial: it was 78.2 billion dollars in 2005, which was more than five times the cost in 1982 (see Lomax and Schrank, 2007).

Different technologies and services have been implemented to alleviate this serious problem. Public transportation plays an important role currently: without public transportation services, the 437 urban areas studied in Lomax and Schrank (2007) would have suffered an additional 541 million hours delay in 2005, a 13% increase costing 10.2 billion dollars. Though public transportation alone contributes significantly to alleviate traffic congestion, it is far from a complete solution to this complex and rapidly worsening problem. We show in this paper that the combination of public transportation with a gasoline tax is a better solution.

We study equilibrium congestion with a continuum of heterogeneous commuters. It enables us to illustrate how a gasoline tax affects the allocation of commuters between public transportation and private vehicles. Furthermore, we show how the reallocation that is resulted from an increased gasoline tax improves the welfare of the society, even helps achieve the social optimality.

We make several assumptions for our model. First, commuters are heterogeneous in the sense that they have different and continuous values of time, and the distribution function of the values is known. Second, we assume that the only public transportation service is bus, which is a reasonable assumption that simplifies the analysis. Third, the objective of the regulator is to minimize the aggregate cost of traffic congestion, which is defined as the sum of the time loss and travel cost (on driving or riding) of all the commuters. Our model is formulated as a large game in which each player (i.e., commuter) makes a binary choice between riding a bus and driving a private vehicle. Schmeidler (1973), Rath (1992), Rauh (2003), and Yu and Zhu (2005) have proved the existence of a Nash equilibrium for such games. In our specific model, we also establish the uniqueness of the Nash equilibrium. It is intuitive that at the equilibrium those commuters whose value of time is less than a threshold value ride the bus, and commuters whose time value is greater than the threshold value drive private vehicles. An interesting result is that at the equilibrium, bus fare may decrease in gasoline price. Therefore, both the income and substitution effects of a gasoline tax would encourage the transfer of commuters from private vehicles to public transport. Consequently, the traffic congestion will be alleviated. Furthermore, Pareto improvement may be achieved by imposing an arbitrarily small gasoline tax. Another result is that if the regulator levies a gasoline tax...
gasoline tax on private vehicles and transfers the tax revenue partly to subsidize bus riders, the social optimality can be achieved. We also argue that highway toll is not an effective instrument from society's point of view since its effects are similar to those of a gasoline tax but it costs commuters' time.

Traffic congestion has long been studied by economists. Vickrey (1969) is one of the first seminal papers on urban traffic congestion. The paper examined equilibrium with a fixed number of identical drivers on a road with a bottleneck. All drivers need to make their decisions on their departure time in order to arrive at work at the same time. Many papers extended Vickrey's simple bottleneck model and offered some solutions to the congestion problem (for example, Arnott et al., 1990, 1993). Similarly, our model also considers commuters choosing their economic behaviors to minimize their value of loss on their commutes. However, commuters are heterogeneous rather than identical in our model. Furthermore, instead of choosing their departure time, commuters choose to ride a bus or drive private vehicles in our model.

Solving traffic congestion problems by congestion taxes has been studied both theoretically and empirically for decades. Keeler and Small (1977) developed a model to estimate the trade-off between the cost of providing urban expressway capacity against the value of travel time to minimize total system costs. They focused their analysis on the Bay Area in California and concluded that peak tolls ranging there. More recently, Kraus (1989), Mayeres and Proost (1997) and Mayeres (2000) evaluated the welfare effects of congestion taxes. All these studies emphasized the importance of congestion taxes in solving traffic congestion problems, but they did not take public transportation into account.

Of those studies that consider traffic congestion problems, most closely related to our paper are Glaister and Lewis (1978) and Shy (1995). Glaister and Lewis (1978) evaluated whether lower fares of subsidized public transportation would encourage usage of public transportation and thus alleviate traffic congestion. The paper showed that "current car traffic and subsidy levels seem broadly correct" but "substantial reallocation of public traffic between times and modes was desirable" in order for social welfare to be improved. By comparison, our work shows that the lower fare of public transportation which is due to a gasoline tax does encourage transfer of commuters from private vehicles to public transportation. As a consequence, the bus fare can help make Pareto improvements and may even achieve social optimality.

Shy (1995) considered the optimal allocation of commuters between private vehicles and public transportation. In his model, commuters made a binary decision between train and car for their commutes. He concluded that social optimality cannot be obtained by adjusting train fare. His conclusions were based on two assumptions: one is that all commuters have the same value of time and the other is that the gasoline price does not affect commuters' choices. By relaxing these two assumptions, our model enables us to analyze the effects of income distribution (indicated by value of time in our model) and a gasoline tax on the transfer of commuters from private vehicles to public transportation. The analysis shows that Pareto improvement can be achieved by imposing an appropriate gasoline tax. Furthermore, social optimality can be obtained if a gasoline tax is imposed and the revenue is transferred to the bus riders as subsidies.

Our work makes two contributions to the literature on the taxation of traffic congestion. First, we introduce Greenshields' model that describes the relationship between travel time and vehicle density. This model is one of the most widely used models in macroscopic transportation planning: it characterizes traffic congestion succinctly and connects traffic congestion to time losses of commuters directly. Second, we incorporate income distribution, public transportation and gasoline taxes into Greenshields' traffic model. This model enables us to analyze the commuter transfer between private vehicles and public transports explicitly. Consequently, we conclude that public transportation is much more effective if a gasoline tax is imposed.

The rest of this paper is organized as follows: Section 2 describes the model and its basic assumptions; Section 3 presents the equilibrium and its properties; Section 4 analyzes the policy implications of the model; Section 5 concludes the paper.

2. The model

2.1. Basic structure

Consider a continuum of commuters of size 1. Each of them uses either public transportation or a private vehicle to commute between the suburbs and a central city with the distance d. To simplify the problem, we assume that the only public transportation is a bus with unlimited capacity and all car drivers drive alone. Let \( \alpha \in [0, 1] \) denote the proportion of car drivers. We use it as the indicator of congestion. Let \( T(\alpha) \) and \( t(\alpha) \) denote the travel time of the bus and a car, respectively. They both only depend on the congestion \( \alpha \). Each commuter has his own value of time \( v \) in the interval \( (0, +\infty) \), and the value of time across commuters is distributed according to a differentiable cumulative distribution function \( F(v) \). We assume that \( F'(v) > 0 \) for all \( v \in (0, \infty) \).

Suppose that the bus operator is regulated such that the marginal effect of a gasoline price on the bus fare is a constant, and the bus fare is also inversely proportional to the size of bus riders. More precisely, the bus fare is formulated as

\[
\phi(q, \alpha) = \frac{b_0 + b_1q}{1 - \alpha},
\]

where \( q \) is gasoline price, \( b_1 \) represents the effect of gasoline price on the bus fare; \( b_0 \) summarizes the effects of all other factors, for example, the fixed cost of the bus, the wage of the driver and so on. For a bus rider with value of time \( v \), his monetary value of loss is given by

\[
L_0(v, \alpha, q) = T(\alpha)v + \phi(q, \alpha),
\]

which is a simple sum of time value loss and the bus fare. Similarly, for a car driver with value of time \( v \), the monetary value of loss is given by

\[
L_1(v, \alpha, q) = t(\alpha)v + (c_0 + c_1q).
\]

The first term of the right hand side in Eq. (3) is just the time value loss and the second term represents the monetary cost of the travel where \( c_1q \) is the spending on gasoline with \( c_1 \) describing the quantity of gasoline consumed by a car per unit time, \( c_0 \) representing all deprecations per unit time.

Each commuter makes a binary choice between riding the bus and driving a private vehicle to minimize his value of loss by comparing \( L_0 \) with \( L_1 \).

Our model can be formulated as a large game. The set of players is \( I = (0, \infty) \), endowed with Borel \( \alpha \)-field \( Z \) and the probability measure \( \mu \) corresponding to the distribution function \( F(\cdot) : \mu(\{0, v\}) = F(v) \) for all \( v > 0 \). All players have the same action set \( A = [0, 1] \), where 0 stands for choosing the public transportation, and 1 for private vehicle. A (pure) strategy profile is a measurable function from \( I \) to \( A \).
strategy profile \( f: I \rightarrow A \), let \( m(f) = \int f_i(y_i(d_i)) \in [0, 1]. \) Let \( I \) be the set of real valued measurable functions defined on \( A \times [0, 1]. \) A game is then a mapping from \( I \) to \( U, \) which assigns a payoff function to each player in \( I. \) In our model this mapping is \( \Gamma : I \rightarrow U, \) where for each \( v \in I, \)

\[
\Gamma(v)(a, m) = -L_v(m, v, q), \quad a \in A, \ m \in [0, 1].
\]

(4)

Note that parameter \( q \) is regarded as a parameter of commuters. Each commuter’s payoff is the opposite of his value loss.

2.2. Assumptions

We impose several assumptions on our model in this subsection. The first two are on the travel time \( t(\alpha) \) and \( T(\alpha). \)

A 1. \( t(\alpha) = 1/(1 - \alpha). \)

This assumption is on the travel time of private vehicles. It is a natural implication of Greenshields’ model (Greenshields, 1935) which is widely used in traffic science to describe traffic congestion (see Gazis, 1974). The model makes the assumption that the velocity \( v \) of a vehicle and concentration \( \rho \) (defined as the number of vehicle per unit distance) are linearly dependent, that is, the difference between the average velocity \( \nu \) and the free flow speed \( v_f \) is proportional to the concentration \( \rho, \) thus,

\[
v = v_f(1 - q \rho),
\]

(5)

where \( q \) is a constant and normally taken to be the jam spacing or reciprocal of the jam density, namely \( q = 1/\rho_m. \) Then this equation is equivalent to

\[
v = v_f(1 - \rho/\rho_m).
\]

(6)

If we rewrite Eq. (6) using parameters of our model, then when the proportion of car drivers is \( \alpha, \) the concentration is \( \alpha/d, \) where \( d \) is commuters’ travel distance defined earlier. Recall that \( v_f = d/t(0) \) is the free flow speed and \( \rho_m = 1/d \) is the jam density. Then Eq. (6) can be rewritten as

\[
d/t(\alpha) = d/t(0) \left( 1 - \alpha/d \right) = d/t(0)(1-\alpha).
\]

Normalizing \( t(0) \) to 1, we get

\[
t(\alpha) = \frac{1}{1-\alpha}.
\]

A 2. \( T(\alpha) = \beta(\alpha)/(1 - \alpha) \) where \( 1 \leq \beta(\alpha) \leq \beta \) and \( \beta'(\alpha) < 0, \) furthermore, \( \beta(\alpha) \) is such that \( T'(\alpha) > 0, \) \( \beta \) is a constant greater than 1.

This assumption is on the travel time of the bus. \( \beta(\alpha) \) describes the relative advantage of driving over riding. When \( \alpha \) is small, \( \beta(\alpha) \) is large, driving has obvious advantage over riding and as \( \alpha \) increases, the advantage becomes less obvious. An extreme case is when the proportion of car drivers is 1, then both the bus and cars stuck, hence driving has no advantage at all. These properties are captured by \( 1 \leq \beta(\alpha) \leq \beta \) and \( \beta'(\alpha) < 0. \) The assumption \( T'(\alpha) > 0 \) states that just like the travel time of private vehicles, the travel time of the bus also increases in the proportion of car drivers.\(^5\)

The following assumption is on the parameters of the model.

A 3. The parameters \( b_0, b_1, c_0, \) and \( c_1 \) satisfy the relationship.

\[
b_0 > c_0, \quad b_1 < c_1
\]

(9)

The assumption states that if all the commuters ride the bus, then both the variable and the fixed costs of the bus that each commuter needs to afford are less than the variable and the fixed costs of a car, respectively.

3. The equilibrium

In this section, we show the existence of a nontrivial Nash equilibrium and discuss some of its properties.

3.1. Existence and uniqueness of an equilibrium

Given a gasoline price \( q, \) a Nash equilibrium is a strategy profile \( g: [0, \infty) \rightarrow \{0, 1\} \) such that for \( \mu \)-almost all \( v, \)

\[
f(\nu(g(v), m(g))) \geq f(\nu(a, m(g))) \quad \text{for all} \quad a \in A = \{0, 1\},
\]

(10)

where \( m(g) = \int g(\nu)\mu(d\nu) = \int g(\nu)df(\nu). \)

An equivalent definition of an equilibrium is a pair \((g, \alpha^*)\) satisfying the following conditions:

(i). For \( \mu \)-almost all \( v > 0, \) a commuter with time value \( v \) employs the strategy \( g(v) \in \{0, 1\} \) which is given by

\[
g(v) = \begin{cases} 1 & \text{if } L_0(v, \alpha^*, q) \geq L_1(v, \alpha^*, q), \\ 0 & \text{otherwise}. \end{cases}
\]

(ii). \( \alpha^* \) satisfies \( \alpha^* = \int g(\nu)df(\nu). \)

If each commuter employs the strategy \( g(v) \) at the equilibrium, then the equilibrium can be described by a new pair \((\alpha^*, v^*)\) which satisfies

\[
T'(\alpha^*)v^* = \frac{b_0 + b_1q}{1-\alpha^*} = \alpha^*(v^* + c_0 + c_1q).
\]

(11)

where \( \alpha^* = 1 - F(v^*). \)

Those commuters whose value of time is greater than \( v^* \) drive cars; those commuters whose value of time is less than \( v^* \) ride the bus. Now, we are ready to state the existence and uniqueness of a nontrivial equilibrium.

Theorem 1. Suppose assumptions A1–A3 hold, then there exists a nontrivial unique Nash equilibrium \((\alpha^*, v^*), \) which satisfies.

\[
v^* (\beta(\alpha^*) - 1) = (c_0 + c_1q) - (b_0 + b_1q).
\]

(12)

where \( \alpha^* \in (0, 1) \) and \( v^* > 0. \)

Proof. See Appendix.\(\blacksquare\)

Remark 1. Notice that \( \alpha = 1 \) is also an equilibrium. When \( \alpha = 1, \) all commuters drive then both \( L_0(v, 1, q) \) and \( L_1(v, 1, q) \) are infinitely large for any commuter. So no one has incentive to change his choice and ride the bus. Hence, \( \alpha = 1 \) is a Nash equilibrium. This trivial equilibrium is unstable because we know from the Case 2 in the proof of Theorem 1 that for all \( \alpha \neq 1, \) there will be commuters of positive measure riding the bus. This means if there is an arbitrary deviation of \( \alpha \) from 1, some commuters will change their choices to ride the bus and the resulted proportion of drivers will deviate from 1. Hereafter we only focus on the nontrivial equilibrium.

\(^4\) The normalization changes the parameters \( b_0, b_1, c_0, c_1. \) Without affecting the results of model, we will keep the notations of these parameters.

\(^5\) The condition \( T'(\alpha) > 0 \) is equivalent to \( \beta'(\alpha)/\beta(\alpha) > 1/(1-\alpha). \) This requires that \( \beta'(\alpha) \) satisfies \( 1/\beta(\alpha) \leq \beta' \) and \( -1/(1-\alpha) < \beta'(\alpha)/\beta(\alpha) < 0, \) As a class of examples, we can take \( \beta(\alpha) = (1-\alpha)^0(\beta - 1) + 1, \) where \( \lambda \in (0, 1). \)
In more general settings, Schmeidler (1973), Rath (1992), Rauh (2003), and Yu and Zhu (2005) have proved the existence of a Nash equilibrium for games with a continuum of players. We go further to show the uniqueness of the (nontrivial) equilibrium by focusing on the specific problem of congestion, and the unique equilibrium enables us to analyze how a gasoline tax affects the allocation of commuters between public transportation and private vehicles.

### 3.2. Properties of the equilibrium

In Subsection 3.1, we proved the existence of a nontrivial equilibrium, and we investigate some properties of the equilibrium in this subsection.

According to Eq. (12) and the relationship \( \alpha^* = 1 - F(v^*) \), the equilibrium proportion of drivers \( \alpha^* \) depends on the distribution function \( F(v) \). A natural question is how the equilibrium proportion of drivers changes with respect to different distribution functions. We answer this question in the following proposition.

**Proposition 1.** Suppose \( F_1(v) \), \( F_2(v) \) are two cumulative distribution functions on \( (0, \infty) \) and \( F_2(\cdot) \) first-order stochastically dominates \( F_1(\cdot) \), then \( v_2^* \geq v_1^* \), where \( v_2^* \) and \( v_1^* \) are threshold time value corresponding to \( F_2(v) \) and \( F_1(v) \), respectively.\(^6\) Consequently, there are more drivers when the distribution function is \( F_2(v) \), i.e., \( \alpha_2^* = 1 - F_2(v_2^*) \geq \alpha_1^* = 1 - F_1(v_1^*) \).

**Proof.** See Appendix.

Suppose there are two groups of commuters A and B with the same size. The cumulative distribution functions of time values in groups A and B are \( F_1(v) \) and \( F_2(v) \), respectively. If \( F_2(v) \) first-order stochastically dominates \( F_1(v) \) then there are more commuters with high value of time in B than in A. The proposition above states that there will be no less than commuters driving private vehicles in group B than in group A. The conclusion is intuitive. Clearly, the higher time value a commuter has, the more he cares about the travel time. As a consequence, the commuters in group B are more eager to save travel time than commuters in group A. Since commuters only care about themselves, we can imagine that there will be a higher proportion of commuters drive in order to save time in group B than in A at the equilibrium.

Based on the assumption that all the commuters have the same value of time, Shy (1995, p.453) concluded that the equilibrium number of passengers who drive private vehicles decreases in the equilibrium value of time \( v \), which is an opposite result to ours. Our result makes sense because we know from assumption A3 that a change in gasoline price has a greater effect on a car driver than it does on a bus rider \((b_1 < c_1)\). Thus a commuter who was indifferent between riding and driving before the increase of gasoline price must ride the bus to avoid higher value loss after the rise of gasoline price.

Analogously, let us look into the effect of increased gasoline price on bus fare. On the one hand, bus fare tends to increase because the marginal effect of gasoline price on the bus fare is a positive constant; this is a direct or "income" effect. On the other hand, bus fare tends to decrease since more commuters ride bus and the fare is inversely proportional to the number of bus riders; this is an indirect or the "substitute" effect. When the indirect effect dominates the direct effect, fare decreases.

We state the effect of gasoline price on the bus fare in the following claim.

**Claim 2.** *Equilibrium bus fare may decrease in gasoline price.*

To see this, recall that \( \phi(q, \alpha^*) = (b_0 + b_1 q)/(1 - \alpha^*) = (b_0 + b_1 q)/F(v^*) \). Then

\[
\frac{db}{dq} = \frac{b_1}{F(v^*)} - \frac{b_0 + b_1 q}{F(v^*)} \frac{dv^*}{dq} = \frac{b_1}{F(v^*)} \left[ 1 - \frac{b_0 + b_1 q}{F(v^*)} \frac{dv^*}{dq} \right].
\]

(14)

Employing Eq. (13), we express Eq. (14) as

\[
\frac{db}{dq} = \frac{b_1}{F(v^*)} \left[ 1 - \frac{b_0 + b_1 q}{F(v^*)} \frac{dv^*}{dq} \cdot \left( -F(v^*) \right) \right].
\]

(15)

Whether the sign of \( \frac{db}{dq} \) is negative depends on the distribution function \( F(\cdot) \) and the parameters of our model. For a set of appropriate parameters of the model, if the second term in the bracket is less than zero, then \( \frac{db}{dq} < 0 \).

This result is interesting because if the bus fare decreases in the gasoline price, then the effect of an increasing gasoline price is so large that there will be a large amount of drivers switch to the bus. As a result, the traffic congestion will be greatly alleviated at the new equilibrium.

### 3.3. Socially optimal proportion of drivers

Let us consider the optimal proportion of drivers that minimizes the aggregate monetary loss to all the commuters. Formally, if \( v \) is a threshold value of time, i.e., those whose value of time is less than \( v \) ride the bus and those whose value of time is greater than \( v \) drive cars. Then the aggregate value of loss to all the commuters is given by

\[
L(v) = \int_0^v \left( F(\alpha) S + \frac{b_0 + b_1 q}{1 - \alpha} \right) dF(S) + \int_v^\infty \left( F(\alpha) S + c_0 + c_1 q \right) dF(S).
\]

(16)

The first term of the right hand side describes aggregate loss to all riders and the second term is aggregate loss to all drivers.
Employing $\alpha = 1 - F(v)$, this equation can be rewritten as

$$L(v) = \frac{\beta(\alpha)}{F(v)} \int_{s}^{E} sD(F(s)) + b_{o} + b_{q} + \int_{s}^{E} sD(F(s)) + c_{1}q + c_{2}(1 - F(v)).$$ \hspace{1cm} (17)

We state a necessary condition of the optimal proportion of drivers in the following proposition.

**Proposition 2.** A necessary condition for $\alpha^*$ to be the optimal proportion of drivers is

$$\left(\beta F(v^0) + \beta - 1\right) \int_{s}^{E} F(s) ds = \psi + \beta \sqrt{\int_{s}^{E} F(s) ds} + c_{0} + c_{1}q,$$ \hspace{1cm} (18)

where $\alpha^* = 1 - F(v^*), \psi$ is the mean value of time for all commuters, i.e., $\psi = \int_{s}^{E} sD(F(s))$.

**Proof.** See Appendix.

Notice that the optimal proportion of drivers does not depend on the bus parameters $b$ and $c$. Therefore, optimality cannot be achieved only by lowering the bus fare to attract more commuters to ride.

Individual decision is not involved in the socially optimal level of congestion $\alpha^*$ we discussed in this subsection. In the following section, we depict the Nash equilibrium of congestion which is based on individual decisions.

### 3.4. Inefficiency of the equilibrium

In this subsection, we investigate the efficiency of the equilibrium. In other words, whether the equilibrium achieves social optimality that has been discussed in Subsection 3.3. For this purpose, we postulate the following assumption.

**A 4.** $\beta\alpha(\alpha)<0$.

Since $\beta(\alpha)$ describes the marginal relative advantage of driving over riding, the assumption states that the marginal relative advantage decreases in the proportion of car drivers. This assumption is only imposed on the proof of **Proposition 3**.

**Proposition 3.** Suppose assumption A4 holds, the equilibrium proportion of car drivers is larger than the optimal proportion of car drivers. More precisely, $\alpha^*<\alpha^*$. The equilibrium is inefficient.

**Proof.** See Appendix.

Now that the equilibrium is inefficient, this result is no surprise as in many situations the presence of externality leads to inefficiency. However, does any policy that improves the welfare of all the commuters exist? We address this question in the following section.

### 4. Policy implications

In this section, we analyze the effects of a gasoline tax and subsidy on the allocation of commuters between riding and driving. Accordingly, we also investigate how these effects affect the social welfare.

#### 4.1. Pareto improving taxes

In this subsection, we illustrate the effects of gasoline tax on the equilibrium and analyze whether the tax can help alleviate traffic congestion.

Suppose that the objective of the policy maker is to minimize the aggregate value of loss to all commuters. Since $v^*$ and $v^0$ both depend on the gasoline price $q$, imposing a gasoline tax changes both of them. According to **Proposition 3**, for any gasoline price $q$, $v^0(q) > v^*(q)$ holds. The inequality implies that social optimality cannot be achieved by solely imposing a gasoline tax. However, if an appropriate gasoline tax is imposed, all the commuters can be better off, i.e., Pareto improving can be achieved. This is stated in the following proposition.

**Proposition 4.** Suppose $q$ and $q(1+\tau)$ are the gasoline prices before and after imposing a gasoline tax $\tau$, respectively. $v^*$ and $v^0$ are the equilibrium value of time corresponding to the gasoline prices $q$ and $q(1+\tau)$, respectively. Then

$$F(v^*) - F(v^0) \geq \max \left\{ \frac{b_{1}\tau t}{b_{0} + b_{1}q}, \frac{c_{1}\tau t}{v^0 + c_{1}q} \right\}$$

is a sufficient condition for $\tau$ to be a Pareto improving gasoline tax.

**Proof.** Please see Appendix.

**Remark 2.** The Pareto improving tax $\tau$ depends on the parameters $b_{0}, b_{1}, c_{0}, c_{1}$ and the distribution function $F(v^0)$, so $\tau$ cannot be solved explicitly without knowing $F(v^0)$. However, we illustrate the existence of such $\tau$ by the following analysis.

Suppose $\tau<1$, which is reasonable. Then the right hand side of the sufficient condition, $\max \left\{ \frac{b_{1}\tau t}{b_{0} + b_{1}q}, \frac{c_{1}\tau t}{v^0 + c_{1}q} \right\}$, is less than 1. Consequently, $F(v^0)-F(v^*) \geq 1$, or equivalently $F(v^0) \geq 2F(v^*)$ is sufficient for $\tau$ to be a Pareto improving tax. According to [Downs (2004, p.185)], in the year 2000, 87.9% of all morning commuters drive private vehicles. Then we can approximate $\alpha^* = 1 - F(v^*)$ by 0.879, accordingly $F(v^0) \approx 0.121$. Then $F(v^0) > 0.121 = 0.242$ will be sufficient for $\tau$ to be a Pareto improving tax.

Even though we assume that the cumulative distribution function $F(v)$ is differentiable in Section 2, the analysis of Pareto improving taxes does not rely on this assumption. This enables us to consider an interesting case that $F(v)$ jumps at $v = v^0$. In this case, according to Eq. (19), an arbitrary increase of $\tau$ will lead to a jump of the left-hand side, so the sufficient condition can be easily satisfied. In other words, imposing an arbitrary small gasoline tax will achieve Pareto improvement for all the commuters. This happens because if $F(v)$ jumps at $v = v^*$ then there is an atom of the distribution function at $v^*$. Those whose value time is $v^*$ drive. However, after the gasoline tax was imposed, they expect that more people will turn to ride the bus so the bus fare will decrease. As a consequence, those whose time value is $v^*$ will ride the bus and this makes $\alpha^*$ drop abruptly. For those who are still driving, even though the gasoline tax increases the cost of driving, the better traffic situation due to the sudden decrease of proportion of car drivers compensates for the increase of gasoline price. So both car drivers and bus riders would be better off.

Analogously, highway toll is not an effective policy. The reasoning is as follows: as a cost-raising policy, highway toll will achieve the same objective as what a gasoline tax would do except that it costs more time of drivers, especially on a road with heavy traffic. The wasted time contradicts the goal of the regulator. So the highway toll is an inferior policy to gasoline taxes in the sense of minimizing the commuters’ value of loss.

#### 4.2. A socially optimal policy

As we showed in Subsection 4.1, a gasoline tax is never efficient, thought it can be Pareto improving. Now we consider another policy: imposing a gasoline tax and return all or part of the tax revenue to the bus riders as subsidies. More precisely, suppose the gasoline tax $\tau$ is defined as before and a proportion $\xi$ of the total tax revenue is transferred to the bus riders as subsidies, then the policy is described by a pair $(\tau, \xi)$. After the policy is imposed, denote $\alpha$ as the new equilibrium proportion of car drivers and $v$ is determined by $\alpha = 1 - F(v)$.

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7 We have one bus and a continuum of cars, therefore we omit the tax revenue on the bus.
Consequently, at this new equilibrium those whose value of time is \( \bar{v} \) are indifferent between driving and riding, that is

\[
\frac{\beta(\bar{v})\bar{v}}{(1-\alpha)} + b_0 + b_1 q(1+\tau) - \zeta R(\tau) = \frac{1}{(1-\alpha)}[\bar{v} + c_0 + c_1 q(1+\tau)],
\]

(20)

where \( R(\tau) \) is tax revenue and it is expressed as

\[
R(\tau) = \frac{c_1}{1-\alpha} q^\tau \alpha.
\]

From Eq. (20), we have

\[
\bar{v} = \frac{1}{\beta(\bar{v}) - 1}[c_0 + c_1 q(1+\tau) - b_0 - b_1 q(1+\tau) + \zeta R(\tau)].
\]

(22)

The aggregate loss of value is

\[
L(\tau, \zeta) = \int_0^\tau T(\bar{v}) + \frac{b_0 + b_1 q(1+\tau)}{1-\alpha} - \frac{\zeta c_1 q^2 \alpha}{(1-\alpha)^2} \right) \, dF(s) + \int_{\bar{v}}^\tau t(\bar{v})(s + c_0 + c_1 q(1+\tau))dF(s),
\]

(23)

where the first term of the right hand side is aggregate loss of value to all bus riders and the second term of the right hand side is aggregate loss of value to all drivers. The regulator’s objective is to minimize \( L \) by implementing the policy \((\tau, \zeta)\). We present in the following proposition that there exists a socially optimal policy \((\tau^*, \zeta^*)\) that minimizes the aggregate loss of value.

**Proposition 5.** There exists a policy \((\tau^*, \zeta^*)\) that minimizes the aggregate loss of value, i.e., the policy \((\tau^*, \zeta^*)\) is socially optimal.

**Proof.** Please see Appendix.

5. Concluding remarks

In this paper, we considered the allocation of a continuum of heterogeneous commuters between public transportation and private vehicles. A unique nontrivial equilibrium was proved to exist and a Pareto improving sufficient condition was provided. Following these results, we also proved the existence of a socially optimal policy that minimizes the aggregate loss of time to all the commuters.

Besides alleviating traffic congestion, the tax policies we provided in this paper can also help reduce gasoline consumption. Consequently, these policies can also reduce the greenhouse gas emissions since automobiles are a source of considerable pollution, including a significant fraction of the total greenhouse gas emissions. To sum up, these policies cannot only improve commuters’ welfare but also contribute to alleviate pollution problems.

Further work can be done by relaxing our assumptions of the model. First, we did not consider the contributions of the other public transports except buses, such as the fixed-rail transit. According to [Downs, 2004, pp.124], “fixed-rail mileage of transit has been a rising share in recent year”. And the commuters on the fixed-rail transit almost do not suffer from the traffic congestion. Therefore, it would be interesting to take into account the contribution of fixed-rail transit. Second, in our paper, the goal of the regulator is to minimize the value of loss to all the commuters and other parties such as the oil industry, and the automobile industry are not included in our model. As a matter of fact, those parties we omitted are some resources of the powerful resistance to the fuel policies. It might be fruitful to incorporate those parties into our model.

## Appendix A

### Proof of Theorem 1

Given a gasoline price \( q \), a commuter with value of time \( v \) drives if and only if

\[
L_0(v, \alpha) \geq L_1(v, \alpha).
\]

(21)

In other words, a commuter with value of time \( v \) chooses his strategy based on whether the ratio \( L_0(v, \alpha)/L_1(v, \alpha) \) is greater or less than 1.

\[
L_0(v, \alpha)/L_1(v, \alpha) = \left( b_0 + b_1 q + \frac{\beta(\alpha)v}{1-\alpha} \right) \left( c_0 + c_1 q + v \right) / (c_0 + c_1 q + v), \alpha \neq 1
\]

(22)

In order to prove the theorem, we define a function

\[
G(\alpha) = \alpha - \int g(v, \alpha)dv(0), 0 \leq \alpha < 1.
\]

(23)

Recall that \( g(v, \alpha) \) takes value either 1 or 0 and it equals 1 if and only if (A.1) holds. Obviously \( G(\alpha) \) is not continuous in \( \alpha \). We need the continuity of \( G(\alpha) \) to prove Proposition 2. To this purpose, we have the following auxiliary result.

**Lemma 1.** \( G(\alpha) \) is a continuous function of \( \alpha \).

**Proof of Lemma 1.** To prove \( G(\alpha) \) is continuous in \( \alpha \), it is sufficient to prove \( \int g(v, \alpha)dv(0) \) is continuous in \( \alpha \).

For all given \( \alpha \in (0, 1) \), \( v(\alpha) = (c_0 + c_1 q - b_0 - b_1 q)/(\beta(\alpha) - 1) \) is the solution to \( L_0(v, \alpha) = L_1(v, \alpha) \). If \( v < v(\alpha) \), then \( g(v, \alpha) = 0 \), i.e., for given congestion \( \alpha \), those commuters whose value of time is less than \( v(\alpha) \) ride the bus. Similarly, if \( v > v(\alpha) \), then \( g(v, \alpha) = 1 \), i.e., for given congestion \( \alpha \), those commuters whose value of time is greater than \( v(\alpha) \) drive private vehicles.

It is clear that \( v(\alpha) \) is continuous in \( \alpha \). Then

\[
\int g(v, \alpha)dv(0) = \int_0^{\infty} g(v, \alpha)dv(0) = \int_0^{\infty} g(v, \alpha)dv(0) = 1 - F(v(\alpha)).
\]

If the congestion is \( \alpha \) and all the commuters made their binary choices based on (A.1), the resulting congestion is \( \int g(v, \alpha)dv(0) \).

Since \( F(v) \) is continuous in \( v \) and \( v(\alpha) \) is continuous in \( \alpha \), \( \int g(v, \alpha)dv(0) \) is continuous in \( \alpha \) and so is \( G(\alpha) \).

Now we consider the following two cases:

**Case 1.** \( \alpha = 0 \).

A commuter with value of time \( v \) compares \( L_0(v, 0) = T(0)v + b_0 + b_1 q \) with \( L_1(v, 0) = T(0)(v + c_0 + c_1 q) \). If \( \frac{b_0 + b_1 q + (\beta(0)v)}{1-\alpha} \leq v \), then \( L_0 \geq L_1 \). That is to say those commuters whose value of time is in the interval \( (\frac{c_0 + c_1 q - b_0 - b_1 q}{\beta(0) - 1}, \infty) \) prefer to drive. Then not all the commuters ride if \( \alpha = 0 \), i.e., \( \int g(v, \alpha)dv(0) > 0 \) and \( G(0) = 0 - \int g(v, \alpha)dv(0) < 0 \).

**Case 2.** \( \alpha \) is sufficiently close to 1.

From (A.2) and the fact that \( \beta(\alpha) > 0 \), we obtain that for all \( \alpha \in (0, 1) \)

\[
L_0(v, \alpha)/L_1(v, \alpha) = \frac{(b_0 + b_1 q + (\beta(\alpha)v)/c_0 + c_1 q + v)}{(b_0 + b_1 q + (\beta(\alpha)v)/c_0 + c_1 q + v)}.
\]

(24)

According to this inequality, if \( 0 < \alpha < (c_0 + c_1 q - b_0 - b_1 q)/(\beta - 1) \), then the right hand side of this inequality is less than 1 and this implies \( L_0/L_1 < 1 \). In other words, those commuters whose value of time are in the
interval \((0, \frac{c-q-b\theta}{\beta_1})\) always prefer to ride. That is for \(v \in (0, 0)\) we have \(g(v, \alpha) = 0\), where \(\theta = (c_0 + c_q + b_0 - b_1 q)/(\beta - \beta_1)\). Accordingly,

\[
\int g(v, \alpha) dF(v) = \int_0^\alpha g(v, \alpha) dF(v) + \int_\alpha^\infty g(v, \alpha) dF(v) = 1 - F(\theta).
\] (A.6)

The equation holds for all \(\alpha \neq 1\), as a result, \(G(\alpha) = \alpha - \int g(v, \alpha) dF(v) \geq 0\) is also true for all \(\alpha \neq 1\). Since \(1 - F(\theta)\) is strictly less than 1 and independent of \(\alpha\), when \(\alpha\) is sufficiently close to 1, there exists an \(\alpha^*\) greater than \(1 - F(\theta)\) such that \(G(\alpha) > 0\).

Now we know that \(G(\alpha)\) is continuous in \(\alpha\), \(G(0) < 0\) and \(G(\alpha) > 0\). Then the intermediate value theorem gives the result that there exists at least an \(\alpha^*\) in the interval \((0, \alpha)\) such that \(G(\alpha) = 0\).

Consequently, \(\alpha^* \in [0, 1)\) holds. Since \(\alpha^* \in [0, 1)\), \(\alpha^*\) is defined by \(\alpha^* = 1 - F(v^*)\).

At this equilibrium \((\alpha^*, v^*)\), the commuters who are indifferent between driving and riding must have time value \(v^*\) such that

\[
T(\alpha^*)v^* + b_0 + b_1 q = t(\alpha^*)(v^* + c_0 + c_q q).
\] (A.7)

Now we turn to the proof of the uniqueness of the equilibrium. The equilibrium condition (A.7) can be rewritten as

\[
v^* (\beta(\alpha^*) - 1) = (c_0 + c_q q) - (b_0 + b_1 q).
\] (A.8)

Define

\[
H(v) = v(\beta(\alpha) - 1) - [(c_0 + c_q q) - (b_0 + b_1 q)].
\] (A.9)

where \(\alpha = 1 - F(v)\). Then \(H(v^*) = 0\). Taking derivative to \(H(v)\) with respect to \(v\), we have

\[
\frac{\partial H(v)}{\partial v} = \beta(\alpha) - 1 + v(1 + \beta(\alpha)^{1/2}) > 0.
\] (A.10)

This equation, together with the fact \(H(v^*) = 0\) guarantees the uniqueness of the nontrivial equilibrium.

Proof of Proposition 1. We prove this proposition by seeking a contradiction. Suppose that \(v_1 < v_1^*\) holds. Since \(F_1(v)\) first-order stochastically dominates \(F_1(v)\), then we have \(F_2(v_1) \leq F_1(v_1^*) \leq F_2(v_1^*)\).

On the other hand, according to the equilibrium condition Eq. (12)

\[
v_1(\beta(\alpha_1^*) - 1) = v_2(\beta(\alpha_2^*) - 1) = (c_0 + c_q q) - (b_0 + b_1 q),
\]

where \(\alpha_1 = 1 - F_1(v_1^*)\); \(\alpha_2 = 1 - F_2(v_1^*)\). Under the assumption \(v_2 > v_1^*\), the inequality \(\beta(\alpha_2^*) > \beta(\alpha_1^*)\) holds. Since \(\beta(\cdot) > 0\), the inequality \(\beta(\alpha_2^*) > \beta(\alpha_1^*)\) implies \(\alpha_2 > \alpha_1^*\). That’s to say \(1 - F_2(v_1^*) > 1 - F_2(v_1^*)\), and accordingly \(F_2(v_2^*) \geq F_2(v_2^*)\). But the inequality \(F_2(v_2^*) > F_2(v_1^*)\) contradicts \(F_2(v_2^*) \leq F_2(v_1^*)\). The contradiction completes the proof of \(v_2 \geq v_1^*\).

Knowing that \(v_2^* > v_1^*\), we can induce \(\beta(\alpha_2^*) \leq \beta(\alpha_1^*)\) from \(v_1^* (\beta(\alpha_1^*) - 1) = v_1^* (\beta(\alpha_2^*) - 1)\). Therefore, we have \(\alpha_1^* \geq \alpha_2^*\) by the monotonicity of \(\beta(\alpha)\).

Proof of Proposition 2. To prove the proposition, we differentiate Eq. (17) with respect to \(v\) and let it be zero. This gives the first-order condition

\[
0 = \frac{dH}{dv} = \frac{\beta F(v) + \beta(\alpha)^{1/2}}{F^2} \int \left[ F'(s) - \frac{\beta(\alpha)^{1/2}}{\beta - \beta(\alpha)^{1/2}} \right] dF(s)
\]

Now, we show that there exists at least one optimal value \(v^* \in (0, \infty)\) such that the first order condition is satisfied. Since \(F'(v)/F(v) > 0\) for all \(v < \infty\), then we only need to analyze the terms in the bracket. When \(v \to 0\),

\[
- (\beta F(v) + \beta) \int \left[ F'(s) - \frac{\beta(\alpha)^{1/2}}{\beta - \beta(\alpha)^{1/2}} \right] dF(s)
\]

\[
= - (\beta(\alpha)^{1/2} - (c_0 + c_q q)),
\] (A.12)

which is negative. On the other hand, when \(v \to \infty\),

\[
- (\beta F(v) + \beta) \int \left[ F'(s) - \frac{\beta(\alpha)^{1/2}}{\beta - \beta(\alpha)^{1/2}} \right] dF(s)
\]

\[
= - (\beta(\alpha)^{1/2} - (c_0 + c_q q)),
\] (A.13)

Eq. (13), together with the fact that \(F'/F^2 > 0\) for all \(v \in [0, \infty)\) guarantees the right hand side of (A.11) is positive when \(v \to \infty\).

According to the intermediate value theorem, (A.12), (A.13) and the fact that \(L(v)\) is a continuous function of \(v\) implies that there exists at least a \(v^* \in (0, \infty)\) such that the first order condition \(\frac{dH}{dv} = 0\) is satisfied. If we check the two endpoints \(v = 0, v = \infty\), we obtain \(L(0^+) = \infty\) and \(L(\infty) = b_0 + b_1 q\) from Eq. (17). Hence, neither \(v = 0\) nor \(v = \infty\) is optimal. Then the optimal value must be in the interval \((0, \infty)\). Of course, it is not necessarily true that each value that satisfies the first order condition is optimal.

To obtain the necessary condition of the optimal value \(v^*\), we further simplify (A.11)

\[
0 = \frac{F^2}{F} \int \left[ \frac{\beta F(v) + \beta}{F^2} \int \left[ F'(s) - \frac{\beta(\alpha)^{1/2}}{\beta - \beta(\alpha)^{1/2}} \right] dF(s) \right]
\]

Integrating by part, we have

\[
0 = \frac{F^2}{F} \int \left[ \frac{\beta F(v) + \beta}{F^2} \int \left[ F'(s) - \frac{\beta(\alpha)^{1/2}}{\beta - \beta(\alpha)^{1/2}} \right] dF(s) \right]
\]

\[
= \frac{F^2}{F} \int \left[ \frac{\beta F(v) + \beta}{F^2} \int \left[ F'(s) - \frac{\beta(\alpha)^{1/2}}{\beta - \beta(\alpha)^{1/2}} \right] dF(s) \right]
\]

Hence, the optimal value \(v^*\) satisfies

\[
(\beta F(v^*) + \beta) \int F'(s) ds = \beta F(v^*) + \beta v^* F^2(v^*) + c_0 + c_q q.
\] (A.15)

Proof of Proposition 3. Since \(\alpha^* = 1 - F(v^*)\), then we only need to show \(v^* \to \infty\). Let \(W(\alpha) = \beta(\alpha)^{1/2}(0)\), then we have \(W(\alpha) = \beta(\alpha)^{1/2}(0)\).

Taking derivative to \(W(\alpha)\) with respect to \(\alpha\) yields

\[
\frac{dW(\alpha)}{d\alpha} = \beta(\alpha)(1-\alpha) - \beta(\alpha) + \beta(\alpha)^{1/2}
\]

As a consequence, we have \(W(\alpha) > 0\) for all \(\alpha \in [0, 1)\). Therefore, the left hand side of Eq. (18) is positive. Then

\[
\vartheta + \beta v^* F^2(v^*) + c_0 + c_q q = \left(\beta F(v^*) + \beta(\alpha)^{1/2} \right) \int F'(s) ds
\]

\[
= (\beta F(v^*) + \beta(\alpha)^{1/2}) v^*.
\] (A.17)
Rearranging the terms, we obtain
\[
(\beta(\alpha) - 1)v^d > v + c_0 + c_1q + \beta \int F^2(v^d) - F(v^d)
\]
\[
\geq v + c_0 + c_1q
\]
\[
> c_0 + c_1q.
\]
(A.18)

The second inequality holds because \(\beta'v^d(F^2(v^d) - F(v^d))\geq 0\), which followed from \(\beta' < 0\) and \(F(v^d) \leq F(v^d)\). Recall the equilibrium condition \((\beta(\alpha) - 1)v^* = c_0 + c_1q - b_0 - b_1q\) and compare it with Eq. (A.18), we obtain
\[
(\beta(\alpha) - 1)v^d > (\beta(\alpha') - 1)v^d.
\]
(A.19)

As shown in Appendix A, function \(H(v) = (\beta(\alpha) - 1)v^d - (c_0 + c_1q) + (b_0 + b_1q)\) increases in \(v\), then Eq. (A.19) implies \(H(v^d) > H(v^*)\) and therefore \(v^* > v^d\).

**Proof of Proposition 4.** When the gasoline tax \(\tau\) is imposed, the new equilibrium value \(v^*\) is greater than \(v^d\) according to the fact that \(dv/dq > 0\). The equilibrium proportion of car drivers \(\alpha^*, v^* = 1 - F(v^*) < \alpha = 1 - F(v^d)\), the congestion is alleviated.

Each individual commuter’s value of loss will be affected by the gasoline tax. Those commuters whose value of time is less than or equal to \(v^d\) ride bus both before and after the gasoline tax was imposed. The welfare change due to the tax is expressed as
\[
G_1(v) = I_0(v, \alpha', q) - I_0(v, \alpha^*, q(1 + \tau)) = \frac{\beta(\alpha')v^d}{1 - \alpha'} - \frac{b_0 + b_1q(1 + \tau)}{1 - \alpha'}[T(\alpha') - T(\alpha^*), v > v^d] \geq 0.
\]
(A.20)

Recall that \(T'(\alpha) > 0\) by Assumption A2, then \(T(\alpha') > T(\alpha^*)\). Consequently \(G_1(v)\) is a strictly increasing function in \(v\). Hence, for the commuters with value of time in the interval \((0, v^*)\), if a commuter with lower value of time is not worse off, then all the commuters with higher value of time are better off. For all commuters with value of time in \((0, v^d)\), a sufficient condition of welfare improving is
\[
G_1(0) = \frac{b_0 + b_1q(1 + \tau)}{1 - \alpha'} \geq 0.
\]
(A.21)

This condition can also be expressed as
\[
\frac{F(v^d) - F(v^*)}{F(v^d)} \geq \frac{b_1q}{b_0 + b_1q}.
\]
(A.22)

Those whose value of time is in the interval \((v^*, \infty)\) drive both before and after the gasoline tax was imposed, and the welfare change is expressed as
\[
G_2(v) = I_0(v, \alpha', q) - I_0(v, \alpha^*, q(1 + \tau)) = \frac{v + c_0 + c_1q}{1 - \alpha'} - \frac{c_0 + c_1q}{1 - \alpha'}[T(\alpha') - T(\alpha^*), v > v^*]
\]
which is an increasing function of \(v\) according to the fact that \(T'(\alpha) > 0\). So all these commuters are better off after the tax is imposed if \(G_2(v^*) \geq 0\) which can be rewritten as
\[
\frac{F(v^* - F(v^d)}{F(v^d)} \geq \frac{c_1q}{v^* + c_0 + c_1q}.
\]
(A.24)

The remaining commuters, whose value of time is in \((v^*, v^d)\), drive cars before the tax is imposed and ride the bus after that. The improvement of their welfare can also be expressed by the function
\[
G_3(v) = \frac{b_1q}{1 - \alpha'} - \frac{c_0 + c_1q}{1 - \alpha'}\left(\frac{\beta(\alpha')v^d}{1 - \alpha'} + b_0 + b_1q(1 + \tau)\right).
\]
(A.25)

Given a gasoline tax \(\tau\), if \(\beta(\alpha')v^d = \frac{1 - \alpha'}{1 - \alpha}\) is a constant, then \(G_3(v)\) is a linear function of \(v\). If commuters with value of time \(v^d\) are not worse off, then all the commuters whose value is in the interval \((v^*, v^d)\) will not be worse off.

To sum up, a sufficient condition for the Pareto improving is
\[
\frac{F(v^d) - F'(v^*)}{F(v^d)} \geq \max\left\{\frac{b_1q}{b_0 + b_1q} - \frac{c_1q}{v^* + c_0 + c_1q}\right\}. \hspace{1cm} (A.26)
\]

According to this condition, whether the gasoline tax is Pareto improving or not depends on the distribution of time value across all commuters.

**Proof of Proposition 5.** The regulator needs to find a pair \((\tau^*, \xi^*)\) solving the problem
\[
in\tau \in [0, \infty] \exists (\tau^*, \xi^*) \exists (\tau^*, \xi^*) = \inf_{\tau \in [0, \infty]} \sup_{\xi \in [0, 1]} L(\tau, \xi)\]
\[
(A.27)
\]
where \(D = \{\xi \in [0, 1] : 0 \leq \xi \leq \tau, \xi R(\tau) \leq b_0 + b_1q(1 + \tau)\}, 0 \leq \xi \leq 1\) and \(0 \leq \tau\) state that the subsidy and the gasoline tax are both nonnegative. \(\xi R(\tau) \leq b_0 + b_1q(1 + \tau)\) describes the nonnegativeness of the bus fare.

In order to complete the proof, we need the following lemma.

**Lemma 2.** When \(\tau\) goes to infinity, the aggregate loss of value goes to infinity uniformly for \(\xi \in [0, 1]\), i.e., \(\lim_{\tau \to \infty} \inf_{\xi \in [0, 1]} L(\tau, \xi) = \infty\).

**Proof.** According to Eq. (22)
\[
\psi = \frac{1}{\beta(\alpha) - 1}[c_0 + c_1q(1 + \tau) - b_0 - b_1q(1 + \tau) + \xi R(\tau)]
\]
\[
> 1 - \frac{1}{\beta(\alpha)}[c_0 + c_1q(1 + \tau) - b_0 - b_1q(1 + \tau)],
\]
both \(\psi\) and \(\alpha\) are functions of the policy parameters \(\xi\) and \(\tau\). From the equation, it is clear that \(\lim_{\tau \to \infty} \inf_{\xi \in [0, 1]} L(\tau, \xi) = \infty\). Then a natural implication for \(\alpha = 1 - F(\psi)\) is \(\lim_{\tau \to \infty} \sup_{\xi \in [0, 1]} L(\tau, \xi) = 0\).

Since the second integral of the right hand side of Eq. (23) is non-negative, we have
\[
in\xi \in [0, 1], L(\tau, \xi) \geq \inf_{\xi \in [0, 1]} \left[\frac{\psi}{\beta(\alpha)} - \frac{b_0 + b_1q(1 + \tau)}{1 - \alpha} - \frac{c_1q}{1 - \alpha}\right] dF(s)
\]
\[
\geq \inf_{\xi \in [0, 1]} \left[\frac{b_0 + b_1q(1 + \tau)}{1 - \alpha} - \frac{c_1q}{1 - \alpha}\right] F(\psi)
\]
\[
\geq \inf_{\xi \in [0, 1]} \left[\frac{b_1q(1 + \tau)}{1 - \alpha} - \frac{c_1q}{1 - \alpha}\right] F(\psi)
\]
\[
\geq \inf_{\xi \in [0, 1]} \left[\frac{b_1q(1 + \tau)}{1 - \alpha} - \frac{c_1q}{1 - \alpha}\right] F(\psi)
\]
\[
\geq \inf_{\xi \in [0, 1]} \left[\frac{b_1q(1 + \tau)}{1 - \alpha} - \frac{c_1q}{1 - \alpha}\right] F(\psi)
\]
\[
\geq \lim_{\tau \to \infty} \left[\frac{b_1q(1 + \tau)}{1 - \alpha} - \frac{c_1q}{1 - \alpha}\right] F(\psi)
\]
\[
= \lim_{\tau \to \infty} b_1q(1 + \tau) = \infty.
\]
(A.29)

The equality holds because \(\lim_{\tau \to \infty} \sup_{\xi \in [0, 1]} L(\tau, \xi) = 0\). Eqs. (A.29) and (A.29) complete the proof.
Lemma 2 implies that for any aggregate value of loss $K$ which is greater than $\inf_{\zeta \in [0,1]} L(\zeta, \tau) > K$. In other words, the solution to $\inf_{\zeta \in [0,1]} L(\zeta, \tau)$ is not in the set $\{(\zeta, \tau) \in D : \tau > \tau_0\}$. Thus we can define $D' = \{(\zeta, \tau) \in D : \tau \leq \tau_0\}$, and conclude that

$$\inf_{(\zeta, \tau) \in D'} L(\zeta, \tau) = \min_{(\zeta, \tau) \in D'} L(\zeta, \tau).$$  \hfill (A.30)

Instead of solving $\inf_{(\zeta, \tau) \in D} L(\zeta, \tau)$, we only need to solve the alternative problem $\min_{(\zeta, \tau) \in D'} L(\zeta, \tau)$. For this problem, since $L(\tau, \zeta)$ is a continuous function defined on the compact set $D'$, the existence of the solution is guaranteed by the Weierstrass Theorem.

References
