Dynamic Decisions under Subjective Expectations: A Structural Analysis

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Abstract

This paper studies dynamic discrete choice models by relaxing the assumption of rational expectations. Agents' expectations about state transition are subjective, unknown, and allowed to differ from their rational counterparts. We show that agents' subjective expectations and preferences can be identified and estimated from observed conditional choice probabilities in both finite and infinite horizon models. Our identification of subjective expectations is nonparametric and can be expressed as a closed-form function of the observed conditional choice probabilities. We estimate the model primitives using maximum likelihood estimation and demonstrate the good performance of estimators using Monte Carlo experiments. Using Panel Study of Income Dynamics (PSID) data, we illustrate our methodology by analyzing women's labor participation. We find evidence of systematic differences between workers' subjective and rational expectations about their income transition.

Keywords: Dynamic discrete choice models, subjective expectations, rational expectations, nonparametric identification, estimation.

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1 Introduction

Decision-making under uncertainty, such as educational choice, labour participation, and occupational choice, is prominent in economics. In the literature, agent choices are modeled as the optimal solution to an expected utility maximization problem, where expected utility is computed using agent expectations about choice-specific future outcomes (e.g., a woman's expectations of household income conditional on her labor participation decision). That is, observed choices are determined by not only the agents' preferences but also their expectations. A central problem in this literature is to infer agent preferences from actual choices observed in the data using the connection between preferences, expectations, and choices. Thus, information on agent expectations is crucial for the inference of preferences from observed choices.

Unfortunately, since the econometrician typically does not observe agent expectations in practice, assumptions are usually imposed. A ubiquitous assumption is that agent expectations are rational, such that agents' subjective expectations about future uncertainty coincide with the distribution of *ex-post* realized outcomes. Such an assumption may be problematic, as Manski (1993a) points out that observed choices can be consistent with multiple combinations of expectations and preferences. Moreover, some recent studies have documented systematic discrepancies between subjective and rational expectations by comparing survey data on agents' subjective expectations with the expectations of their objective counterparts (see e.g., Heimer, Myrseth, and Schoenle (2018) and Cruces, Perez-Truglia, and Tetaz (2013), among others). Not surprisingly, violation of the rational expectations assumption may induce biased estimation of agent preferences and misleading counterfactual results. A dominating solution in the literature is to solicit subjective expectations (see Manski (2004) for a review) and to study agent decisions under the solicited expectations (see e.g., Van der Klaauw (2012)). Nevertheless, the availability of such surveyed expectations is very limited. For some historical datasets, especially, it is impossible to collect agents' subjective expectations.

In the existing literature, little is known about what can be achieved if there are neither solicited subjective expectations nor a known link between subjective expectations and some observables, such as the assumption of rational or myopic expectations. We provide a first positive result by showing that we can recover both agents' preferences and their subjective expectations, using their observed choices and some assumptions described later. Specifically, we consider a standard dynamic discrete choice (DDC) model where agents may have subjective expectations about the law of motion for state variables, which are unknown to the econometrician. We provide a unified identification strategy for identifying agent preferences, together with their subjective expectations in both finite and infinite horizon models. The identifying power is mainly from exclusion restrictions to expectations and flow utility provided by either time or combination of time and an additional state variable in finite horizon and an additional state variable in infinite horizon. Our identification results apply to both homogenous and heterogenous expectations.

We show that when agents' subjective expectations are homogenous, these expectations can be identified and estimated from the observed conditional choice probabilities

(CCPs) in both finite-horizon and infinite-horizon models. Based on the insight of under-identification results, e.g., Rust (1994) and Magnac and Thesmar (2002), we address identification of DDC models by assuming that the discount factor and the distribution of agents' unobserved preference shocks are known. Our methodology then identifies agents' subjective expectations on state transition as a closed-form solution to a set of nonlinear moment conditions that are induced from Bellman equations using the insight in Hotz and Miller (1993). The key identification assumptions in both finite and infinite horizon models are that subjective expectations and preferences are time-invariant, exclusion restrictions exist, and subjective expectations are partially known to the econometrician.

In the finite horizon framework, we first use time as an exclusion restriction to both subjective expectations and flow utility to identify both agents' expectations and their preferences. Intuitively, if both the utility function and expectations are time-invariant, CCPs for a given state would only change over time because of proximity to the terminal period. The difference in CCPs at two different time periods under the same state must be attributed to different continuation values in the next period (due to proximity to the terminal period) aggregated over agent subjective expectations. Considering that expectations do not change over time, we can control the impact of continuation values via their recursive relationship. It would then be possible to disentangle expectations from continuation values with multiple consecutive time periods of data, given sufficient variation in expectations across states and actions. Time, essentially, serves as an exclusion restriction that shifts CCPs without altering preferences or expectations. We show that identification requires data with a sufficiently large number of periods— a number that is at least the cardinality of the state variable's support. This requirement may limit the applicability of our identification results in some empirical studies where the support of the state variable is large. Nevertheless, we prove that the required number of periods would be greatly reduced if there exists an additional state variable with known transition which is an additional exclusion restriction to expectations, i.e., it is independent of subjective expectations. If the cardinality of the additional state variable is not fewer than that of the state variable with subjective expectations, four consecutive periods of data is sufficient for identification. In addition to the existence of additional exclusion restriction to expectations, identification requires that subjective expectations to be known (or normalized) for an action or for an action given a state. Such a requirement is due to nature of DDC models, where CCP for any action depends on choice-specific values of all actions relative to that of a reference action, and these relative choice-specific values further rely on the relative subjective expectations. Consequently, a normalization is necessary to fully identify subjective expectations for all actions and states. It is worth mentioning that this additional state variable does not need to be exclusion restriction for flow utility besides stationarity of preference for identifying subjective expectations.

In infinite-horizon models, the stationarity of CCPs rules out the possibility of using time as an exclusion restriction. However, we can use an additional state variable as the exclusion restriction and employ an identification strategy similar to that of the finite horizon case. Specifically, this state variable is an exclusion restriction to subjective expectations, i.e., it evolves independently from expectations; its transition needs to be known to the econometrician; it is also an exclusion restriction to flow utility. Given the exclusion restriction, the identification argument is similar to that of the finite horizon model. Specifically, the CCPs for different values of the additional state variable reveal information about agent flow utility, expectations, and *ex-ante* value function. The impact of utility on the CCPs can be controlled by the exclusion restriction, and the *ex-ante* value function can be identified as the fixed-point solution to a contraction mapping, given that both preferences and subjective expectations are normalized (known) for a reference action. Identification requires the cardinality of the exclusive variable to be at least the cardinality of the state variable of subjective expectation.

Our identification strategies also apply to DDC models wherein agents hold heterogenous subjective expectations and/or preferences. Assuming that agents are classified into finite unobserved types with agents of the same type having homogenous expectations and/or preferences, we prove that type-specific CCPs can be nonparametrically identified. This step uses recently developed methodologies in measurement error, e.g., Hu (2008). Once the type-specific CCPs are recovered, one can apply the identification results developed for homogenous expectations to identify the type-specific subjective expectations and/or preferences. Not surprisingly, identification under heterogenous expectations may require more periods of data than that with homogenous expectations.

We propose a maximum likelihood estimator for the model primitives, including agents' preferences and subjective expectations in both finite and infinite horizon cases. Our Monte Carlo experiments show that the proposed estimator performs well with moderate sample sizes, and that performance is maintained when the data are generated under rational expectations. Furthermore, we find that imposing rational expectations leads to inconsistent estimation of payoff primitives if the data are generated from subjective expectations that differ from their objective counterparts.

We illustrate our methodology by analyzing women's labor participation using Panel Study of Income Dynamics (PSID) data. In order to decide whether to join the labor force or stay at home, the household needs to perceive how the wife's labor force status would affect future household income. We discretize household income into three groups: low, medium, and high. Our estimation results reveal clear discrepancies between subjective expectations about state transitions and their objective counterparts conditional on both working and not working. We find that households with a non-working wife are overly pessimistic about their income transitions, while those with a working wife are less so. In addition, we show that agents have "asymmetric" expectations about income transitions. Conditional on not working, agents with medium income believe that income will remain at medium with very high probability. However, agents with high income are more pessimistic and believe that income will drop to medium with almost certainty, compared to the objective probability of income drop being merely 0.24. We test the hypothesis that agents hold rational expectation and decisively reject the null hypothesis, implying that agents do not hold rational expectations about income transition. We further simulate agent CCPs under both subjective expectations and rational expectations. Our results suggest that having subjective expectations increases the probability of working compared to rational expectations. The effects are heterogenous across income levels: women with low income are more likely to work due to their subjective expectations than women with medium or high income.

This paper is related to the rapidly growing literature on subjective expectations. Relaxing rational expectations in DDC models, or decision models in general, is of both theoretical and empirical importance. Manski (2004) advocates using data on subjective expectations in empirical decision models. In related literature, a substantial effort has been invested into collecting data regarding agents' subjective expectations, so that the econometrician can employ these subjective expectations directly to study agents' behaviors under uncertainty. For example, Van der Klaauw and Wolpin (2008) studies social security and savings using a DDC model for which agents' subjective expectations about their own retirement age, anticipated longevity, and future changes in the Social Security program come from surveys. Zafar (2011, 2013) study schooling choice using survey data on students' subjective expectations. Wang (2014) uses individuals' subjective longevity expectations to explain adult smokers' decision to smoke in a framework of dynamic discrete choice. Acknowledging the scarcity of expectations data, we take a distinctive approach from this literature and focus on inferring agents' subjective expectations from their observed choices.

This paper also contributes to a growing literature on the identification of dynamic discrete choice models. Rust (1994) provides some non-identification results for the infinite-horizon case. Magnac and Thesmar (2002) further determines the exact degree of under-identification and explore the identifying power of some exclusion restrictions. Kasahara and Shimotsu (2009) and Hu and Shum (2012) consider identification of DDC models with unobserved heterogeneity/state variables. Abbring and Daljord (2018) use an exclusion restriction to identify the discount factor. Abbring (2010) provides an excellent review on identification of DDC models. The assumption of rational expectations is imposed and plays an important role for identification in all these papers. We relax the assumption of rational expectations and propose an original argument that not been used in other contexts to identify agents' subjective expectations. The estimated subjective expectations can be used to test the widely imposed assumption of rational expectations.

Our paper is also related to Aguirregabiria and Magesan (2019), which studies identification and estimation of dynamic games where strategic interaction among players are crucial. In contrast to the existing literature that assumes Nash equilibrium, Aguirregabiria and Magesan (2019) identifies players' payoffs and their expectations about rivals' behaviors while allowing those expectations to be inconsistent with their equilibrium counterparts. The key identification condition in Aguirregabiria and Magesan (2019) is the existence of an exclusion restriction and partially unbiased expectations, similar to the identification conditions in our paper. However, Aguirregabiria and Magesan (2019) differs from our paper in that players in their model still have rational expectations about state transitions, and it is unclear how their identification strategy would apply to our setting. Moreover, we show that both time and an additional state variable provide exclusion restrictions for identification, while Aguirregabiria and Magesan (2019) only explores the identification power of state variables.

The remainder of this paper is organized as follows. Section 2 presents dynamic discrete choice models with subjective expectations. Section 3 proposes identification results for the finite horizon case, where time is used as an exclusion restriction. Section 4 presents identification strategies relying on exclusion restrictions provided by an additional state variable. Section 5 extends identification to a model with heterogeneous expectations and/or preferences. Section 6 discusses estimation and provides Monte Carlo evidence. Section 7 studies women's labor participation by applying our method to PSID data. Section 8 concludes. Proofs are presented in the Appendix.

2 DDC Model with Subjective Expectations

In this section, we describe a DDC model where rational expectations of agents are relaxed and present basic assumptions.

In each discrete time period t, a single agent chooses one action a_t from a finite set of actions, $\mathcal{A} = \{1, \cdots, K\}, K \geq 2$, to maximize her expected utility. The utility-relevant state variables in period t consist of two parts, x_t and ϵ_t, where x_t is the state variable observed by the econometrician, and ϵ_t is an unobserved vector of choice-specific shocks, i.e., $\epsilon_t = (\epsilon_t(1), \cdots, \epsilon_t(K))$. We assume that the observed state variable x_t is discrete and takes values in $\mathcal{X} \equiv \{1, \cdots, J\}, J \geq 2$. Both state variables are realized to the agent at the beginning of period t. The agent then makes choice a_t and obtains per-period utility $u(x_t, a_t, \epsilon_t)$. There is uncertainty regarding future states, which is governed by nature or an exogenous mechanism and is assumed to be a Markov process. Specifically, given the current state (x, ϵ) and agent choice a, the state variables in the next period (x', ϵ') are determined by the transition function $f(x', \epsilon'|x, \epsilon, a)$, for which we suppress period subscripts and use prime to represent the next period for ease of notation. Following the existing literature, we impose the following assumption on the transition function.

Assumption 1 (Conditional Independence) *(a). The observed and unobserved state variables evolve independently conditional on x and a. That is,*

$$f(x', \epsilon'|x, a, \epsilon) = f(x'|x, a)f(\epsilon'|x, a, \epsilon).$$

(b). The unobserved state variables across time periods and actions are i.i.d. draws from the mean zero type-I extreme value distribution.[1] *That is,*

$$f(\epsilon'|x, a, \epsilon) = f(\epsilon').$$

Under Assumption 1, the transition process can be simplified as

$$f(x', \epsilon'|x, a, \epsilon) = f(x'|x, a)f(\epsilon').$$

Because the agent is forward-looking and her choice involves intertemporal optimization, her expectations of the state transition plays an essential role in the her decision-making

[1]The assumption of type-I extreme value distribution is for ease of illustration. As long as the distribution is known and absolutely continuous, our identification argument holds.

process. Let $s(x'|x,a)$ be the agent's (subjective) expectation about the transition of the observed state. In the literature of DDC models, a ubiquitous assumption is that agents have perfect expectations or rational expectations, i.e., $s(x'|x,a) = f(x'|x,a)$ for all x', x and a, and this is crucial for identifying and estimating DDC models (e.g., see Magnac and Thesmar (2002)). Unfortunately, this assumption is very restrictive. Manski (1993b) points out that even a learning process may not justify rational expectations if the law of motion changes between the two cohorts due to some macro-level shocks, or the earlier cohort's history cannot be fully observed. Moreover, the recent literature documented violations of rational expectations by comparing survey data on agents' subjective expectations with their objective counterparts and examining the significant impact of violations on agent choices. For example, Heimer, Myrseth, and Schoenle (2018) shows that surveyed mortality expectations over the life cycle substantially differ from actuarial statistics from the Social Security Administration; this discrepancy leads the young to under-save by 30% and also causes retirees to draw down on their assets 15% more slowly than ideal. Cruces, Perez-Truglia, and Tetaz (2013) provide evidence of agents' biased perception of the income distribution. In the literature on structural models of oligopoly competition, firms may also have biased expectations about model primitives. See Aguirregabiria and Jeon (2018) for a recent review of structural models of oligopoly competition where firms have biased expectations about the behavior of their rivals or other model primitives.

Motivated by the above theoretical argument and empirical evidence, we relax the assumption of rational expectations in our model. In what follows, we describe the agent's problem in a general framework allowing for subjective expectations, lay out some basic assumptions, and characterize the agent's optimal decision.

In each discrete period, the agent's problem is to decide the action that maximizes her expected life-time utility, based on her subjective expectations about the future evolution of the state variable. The optimization problem is characterized as

$$\max_{a_t \in \mathcal{A}} \sum_{\tau=t,t+1,\ldots} \beta^{\tau-t} E\big[u(x_\tau, a_\tau, \epsilon_\tau) | x_t, a_t, \epsilon_t\big],$$

where $\beta \in [0,1)$ is the discount factor, $u(x_\tau, a_\tau, \epsilon_\tau)$ is the flow utility, and the expectation is taken over all future actions and states, based on the agent's subjective expectations $s(x'|x,a)$. These expectations are a complete set of conditional probabilities that satisfy the following properties:

Assumption 2 (valid stationary expectation) *Agents' subjective expectations about the transition probabilities of the observed state variable satisfy the following conditions:*

(a) $\sum_{x' \in \mathcal{X}} s(x'|x,a) = 1$ *and* $s(x'|x,a) \geq 0$ *for any* $x \in \mathcal{X}$ *and* $a \in \mathcal{A}$.

(b) $s(x'|x,a)$ *is time-invariant.*

Assumption 2(a) states some minimum requirements for subjective expectations as probabilities. Part (b) restricts subjective expectations to be stationary and rules out the possibility that agents update their expectations about the transition through learning. There are several reasons why subjective expectations could be stationary and differ from their objective counterparts. First, suppose agents do update their expectations based on some historical information. They may not be able to learn the true objective state transitions, even in the steady state if the prior is not in the support of the true transitions. For example, Esponda and Pouzo (2015) theoretically justifies under which conditions agents can hold biased expectations in the steady state. Second, it is possible that agents only update their expectations after accumulating sufficient evidence, rather than update in every period (see e.g., Lee (2012)). In such a case, it would be reasonable to assume that expectations remain constant for some periods.

Assumption 2(b) is an approximation of agent expectations about state transitions. It might be strong and unrealistic in some applications, where learning should be incorporated. However, modeling learning can be very challenging. Empirically, it is unclear what information agents incorporate into their learning process, e.g., agents may learn from their own experience, their cohort's (Manski (1993b)), or both. An incomplete understanding of the sources of the underlying learning process complicates the framework in both theoretical and empirical analyses. In summary, modeling agents' learning about their subjective expectations is still an open question and we leave it to future work.

It is worth noting that assuming stationary subjective expectations is less restrictive than assuming stationary rational expectations, and our model with subjective expectations nests the existing DDC model with stationary rational expectations as a special case. Note that whether agents have subjective expectations or rational expectations, which capture agent perception regarding the true evolution, transitions of the state variable are still governed by the objective distribution $f(x'|x,a)$. On the other hand, agents' optimal behaviors would depend on their perception of the objective transition.

Next, we present the following assumption concerning agent preferences following the existing literature (e.g., Rust (1987)).

Assumption 3 (additively separable and stationary preference) *The flow utility is time-invariant.*[2] *The unobserved state is assumed to enter the preference additively and separably, i.e.,* $u(x,a,\epsilon) = u(x,a) + \epsilon(a) \equiv u_a(x) + \epsilon(a)$ *for any* $a \in \mathcal{A}$.

This stationarity and additive separability of agent utility imposed in Assumption 3 is widely used in the literature. Consequently, we can represent the agent's optimal choice a_t, which depends on the state, x_t and ϵ_t, in period t as

$$a_t = \arg\max_{a\in\mathcal{A}} \Big\{ u_a(x_t) + \epsilon(a) + \beta \sum\nolimits_{x'\in\mathcal{X}} V_{t+1}(x') s(x'|x,a) \Big\}.$$

where $V_{t+1}(x)$ is the *ex-ante* or continuation value function in period $t+1$ and satisfies the following equation:

$$V_t(x) = \max_{a\in\mathcal{A}} \Big\{ u_a(x_t) + \epsilon(a) + \beta \sum\nolimits_{x'\in\mathcal{X}} V_{t+1}(x') s(x'|x,a) \Big\}. \tag{1}$$

[2]With this assumption, time can be used an exclusion restriction, as in Bajari, Chu, Nekipelov, and Park (2016).

In the finite horizon setting, the model can be solved using backward induction, starting from the terminal period. This requires a specification on the continuation value at the terminal period. In the existing literature, the continuation value at the terminal period could be zero or nonzero, depending on the empirical context. In the infinite horizon setting, stationarity restricts the value function to be a fixed point of a contraction mapping (see e.g., Aguirregabiria and Mira (2010) for details).

Following the existing literature, we characterize agents' optimal behavior using a whole set of probabilities (CCPs) that each action $i \in \mathcal{A}$ is chosen conditional on the observed state in period t, denoted as $p_{t,i}(x)$. Assuming that ϵ is distributed according to type-I extreme value distribution with mean zero, the agent's optimal behavior can be characterized in the following

$$p_{t,i}(x) = \frac{\exp(v_{t,i}(x))}{\sum_{a \in \mathcal{A}} \exp(v_{t,a}(x))}, \tag{2}$$

where $v_{t,a}(x)$ is the choice-specific value function for action a, conditional on state x,

$$v_{t,a}(x) \equiv u_a(x) + \beta \sum_{x' \in \mathcal{X}} V_{t+1}(x') s(x'|x, a).$$

Equation (2) allows us to obtain the log ratio of CCPs at time t between any two actions for a given state x:

$$\begin{aligned} \log\left(\frac{p_{t,i}(x)}{p_{t,K}(x)}\right) &= v_{t,i}(x) - v_{t,K}(x) \\ &= \big[u_i(x) - u_K(x)\big] + \beta \sum_{x' \in \mathcal{X}} V_{t+1}(x') \big[s(x'|x, i) - s(x'|x, K)\big]. \end{aligned} \tag{3}$$

This log ratio of CCPs reveals information about the relative flow utility, *ex-ante* values, and the difference in subjective expectations between two different actions.

3 Identification with Time Variations in CCPs

This section provides sufficient conditions under which agents' subjective expectations are uniquely determined by their CCPs in the finite horizon framework without making any restrictions on the continuation value in the terminal period. The main idea of identification is to build a relationship between observed CCPs and unknown subjective expectations by exploring the variation of CCPs over time. Specifically, time is treated as an exclusion restriction because it shifts neither the flow utility nor the expectations but does affect CCPs due to proximity to the terminal period.

Suppose we observe data $\{a_t, x_t\}$, where $t = 1, ..., T$, and T is not necessarily the terminal period of agent decision. The log ratio of CCPs for action $i \in \mathcal{A}$ over K $(i \neq K)$

in period t for state x can be represented in the following matrix fashion.

$$\xi_{t,i,K}(x) \equiv \log\left(\frac{p_{t,i}(x)}{p_{t,K}(x)}\right) = u_i(x) - u_K(x) + \beta\big[S_i(x) - S_K(x)\big]\boldsymbol{V}_{t+1}, t = 1, 2, \cdots, T, \quad (4)$$

where $S_a(x) \equiv [s(x' = 1|x, a), ..., s(x' = J-1|x, a)], \forall a \in \mathcal{A}$, is a $1 \times (J-1)$ vector capturing expectations associated with action a and state x but excluding the element $s(x' = J|x, a)$ because they sum up to one; the *ex-ante* value function vector $\boldsymbol{V}_{t+1}$, thus, is constructed as a $(J-1) \times 1$ vector that consists of relative values using J as a reference state, i.e., $\boldsymbol{V}_{t+1} \equiv [V_{t+1}(x = 1) - V_{t+1}(J), ..., V_{t+1}(x = J-1) - V_{t+1}(J)]'$.

From equation (4), log ratios of CCP $\xi_{t,i,K}(x)$ are determined by three components: the difference in utilities between action i and K, $u_i(x) - u_K(x)$; the continuation value $\boldsymbol{V}_{t+1}$ discounted by β; and the subjective expectations conditional on action i relative to action K, $S_i(x) - S_K(x)$. We assume discount factor β is known because it is not the focus of our paper.[3] To separately recover those components from the observed CCPs, we rely on exclusion restrictions that shift the log ratios without affecting the difference in flow utility, so that the variation in the log ratio of CCPs only reveals information about agent expectations once we control for the *ex-ante* value function. Time can serve as such an exclusion restriction because agent preferences and expectations are time-invariant and we can control the impact of the flow utility on the CCPs by taking the first difference of the log ratio of CCPs along the time dimension for any given state. That is,

$$\Delta\xi_{t,i,K}(x) \equiv \xi_{t,i,K}(x) - \xi_{t-1,i,K}(x) = \beta\big[S_i(x) - S_K(x)\big]\Delta\boldsymbol{V}_{t+1}, t = 2, 3, \cdots, T-1, (5)$$

where $\Delta\boldsymbol{V}_{t+1} \equiv \boldsymbol{V}_{t+1} - \boldsymbol{V}_t$ captures the first difference of the relative *ex-ante* value functions. To proceed, we stack equation (5) and collect all $J-1$ equations for $x = 1, 2, \cdots, J-1$ to obtain the following matrix representation of the equation above,

$$\boldsymbol{\Delta\xi}_{t,i,K} = \beta\big[\boldsymbol{S}_i - \boldsymbol{S}_K\big]\boldsymbol{\Delta V}_{t+1}, \quad (6)$$

where $\boldsymbol{\Delta\xi}_{t,i,K} \equiv [\Delta\xi_{t,i,K}(1), ..., \Delta\xi_{t,i,K}(J-1)]'$ collects the first difference of the log ratio of CCPs for $J-1$ values of x and $\boldsymbol{S}_a \equiv [S_a(x = 1), ..., S_a(x = J-1)]'$ is a $(J-1) \times (J-1)$ matrix that stacks the expectations (expectation matrix) associated with action a for $x \in \{1, ..., J-1\}$.

Equation (6) summarizes restrictions implied by the model on CCPs for actions i and K. These restrictions are insufficient for us to identify $\boldsymbol{S}_i$ and $\boldsymbol{S}_K$ because $\boldsymbol{\Delta V}_{t+1}$ is unknown. Considering that subjective expectations are time-invariant, we can control the impact of continuation values on CCPs via its recursive relationship and then disentangle subjective expectations from continuation values. For this purpose, we obtain extra restrictions for the model primitives by exploring the recursive relation between

[3] We refer to Magnac and Thesmar (2002) and Abbring and Daljord (2018) for the identification of the discount factor β.

continuation value functions by backward induction. That is,

$$\boldsymbol{V}_t = -\log \boldsymbol{p}_{t,K} + \boldsymbol{u}_K + \widetilde{\boldsymbol{S}}_K \boldsymbol{V}_{t+1}, \tag{7}$$

where $\widetilde{\boldsymbol{S}}_K$ is defined as a $(J-1)\times(J-1)$ matrix with its j-th row $(j = 1, 2, \cdots, J-1)$ being the expectations of an agent taking action $a = K$ and the current state is $x = j$ relative to that of state $x = J$, i.e., $S_K(j) - S_K(J)$. The vector of log CCPs, $\log \boldsymbol{p}_{t,K}$, and of flow utility, $\boldsymbol{u}_K$, are defined analogously. $\widetilde{\boldsymbol{S}}_K$ is necessary because the vector of the *ex-ante* value function is formulated as the relative *ex-ante* value of state x with respect to the reference state J, i.e., $V_t(x) - V_t(J)$.

Intuitively, equation (7) indicates that the *ex-ante* value from optimal behaviors can be expressed as the overall value of choosing action K, $\boldsymbol{u}_K + \widetilde{\boldsymbol{S}}_K \boldsymbol{V}_{t+1}$, and a non-negative adjustment term, $-\log \boldsymbol{p}_{t,K}$, which adjusts for the fact that K may not be the optimal choice. This adjustment term goes to zero as the probability of selecting K goes to one. Note the recursive relation in equation (7) holds for all the choices and is derived from agent's optimization condition via backward induction, so it requires no additional assumptions but provides further moment conditions besides equation (5) regarding model primitives.

We take advantage of the time-invariant utility again and obtain a recursive relationship for the first difference of the *ex-ante* value function $\boldsymbol{V}_t$,

$$\Delta \boldsymbol{V}_t \;\; = \;\; -\Delta \log \boldsymbol{p}_{t,K} + \beta \widetilde{\boldsymbol{S}}_K \boldsymbol{\Delta V}_{t+1}, \tag{8}$$

Equations (6) and (8) summarize all the restrictions implied by the model in any consecutive periods t and $t-1$. Combining the two equations allows us to disentangle $\boldsymbol{S}_i - \boldsymbol{S}_K$ and $\boldsymbol{\Delta V}_{t+1}$. Specifically, in the first step, we separate expectations $\boldsymbol{S}_i - \boldsymbol{S}_K$ from *ex-ante* values $\boldsymbol{\Delta V}_{t+1}$ using equation (6). The separation enables us to represent the *ex-ante* value as a function of the expectations. For this purpose, we make the following assumption.

Assumption 4 *There exists one action i, $i \neq K$ such that the $(J-1)\times(J-1)$ expectation matrix $\boldsymbol{S}_i - \boldsymbol{S}_K$ is full rank.*

To better understand this full rank condition, we explore the restrictions that are imposed on the model by this assumption. If $J = 2$, i.e., $x \in \{1, 2\}$, the full rank condition is simplified as $s(x' = 1|x = 1, i) \neq s(x' = 1|x = 1, K)$, implying that the two expectation vectors are not the same. In case of $J \geq 3$, the full rank condition restricts the $J-1$ expectation differences $S_i(x) - S_K(x)$, $x = 1, ..., J-1$, to be linearly independent.

The fact that the full rank condition is required can be justified as follows. To disentangle subjective expectations from continuation values in equation (6), we have to control the impact continuation values on CCPs via its recursive relationship, equation (8). Because there is no time variation in subjective expectations, it is clear from (6) that sufficient variation in subjective expectations across states and actions is crucial for us to control the impact of continuation values on CCPs.

Under Assumption 4, the difference of *ex-ante* value functions $\boldsymbol{\Delta V}_{t+1}$ can be expressed explicitly in a closed form expression:

$$\boldsymbol{\Delta V}_{t+1} = \beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\boldsymbol{\Delta \xi}_{t,i,K}, \;\; t = 2, \cdots, T. \tag{9}$$

The recursive relationship in equation (8), together with the closed-form expression above, allows us to obtain the moment condition with subjective expectations being the only unknowns. That is,

$$\beta^{-1}\big[\boldsymbol{S}_i - \boldsymbol{S}_K\big]^{-1}\boldsymbol{\Delta \xi}_{t-1,i,K} = -\Delta \log \boldsymbol{p}_{t,K} + \widetilde{\boldsymbol{S}}_K\big[\boldsymbol{S}_i - \boldsymbol{S}_K\big]^{-1}\boldsymbol{\Delta \xi}_{t,i,K}, \;\; t = 3, \cdots, T. \tag{10}$$

Stacking all moment conditions for time $t = 3, \cdots, T$ leads to the following equation

$$\Big[\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}, \quad -\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Big]\Delta\boldsymbol{\xi}_{i,K} = \Delta \log \boldsymbol{p}_K, \tag{11}$$

where $\Delta \log \boldsymbol{p}_K \equiv \Big[\Delta \log \boldsymbol{p}_{2,K}, \Delta \log \boldsymbol{p}_{3,K}, ..., \Delta \log \boldsymbol{p}_{T-1,K}\Big]$ captures all first differences of log CCPs over time and $\Delta\boldsymbol{\xi}_{i,K} \equiv \begin{bmatrix} \Delta\boldsymbol{\xi}^1_{i,K} \\ \Delta\boldsymbol{\xi}^2_{i,K} \end{bmatrix} \equiv \begin{bmatrix} \Delta\boldsymbol{\xi}_{3,i,K} & \Delta\boldsymbol{\xi}_{4,i,K} & \cdots & \Delta\boldsymbol{\xi}_{T,i,K} \\ \Delta\boldsymbol{\xi}_{2,i,K} & \Delta\boldsymbol{\xi}_{3,i,K} & \cdots & \Delta\boldsymbol{\xi}_{T-1,i,K} \end{bmatrix}$ collects all log ratios of CCPs over time. The moment conditions (11) provide a direct link between the first differences of CCP log ratios and the first differences of subjective expectations through a nonlinear system of equations. Identification of expectations relies on the variation of CCP log ratios over time, which is summarized in matrix $\Delta\boldsymbol{\xi}_{i,K}$ with dimensions $(2J-2)$ by $(T-2)$. To recover the unknown subjective expectations from equation (11), sufficient variation in CCP log ratios across time is required. Such requirements are satisfied under the following assumption.

Assumption 5A *(a). The number of periods is sufficiently large, $T \geq 2J$. (b). The matrix $\Delta\boldsymbol{\xi}_{i,K}$ is of full row rank.*

This assumption is imposed on the observed CCPs and is therefore directly testable. Equation (11) is a linear system regarding each row of the two unknown matrices $\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}$ and $\boldsymbol{S}_i - \boldsymbol{S}_K$. Note that each row of the unknown matrix has $J-1$ parameters while each cell of three consecutive periods of data (e.g., $\Delta\boldsymbol{\xi}_{t,i,K}, \Delta\boldsymbol{\xi}_{t-1,i,K}$, and $\Delta \log \boldsymbol{p}_{t,K}$ involve periods $t-1, t$, and $t+1$) provides *one* restriction to the unknown row parameters. Solving this linear system requires (1) the number of restrictions $(T-2)$ is no less than the number of parameters $2(J-1)$, which implies $T \geq 2J$; and (2) the matrix of data $\Delta\boldsymbol{\xi}_{i,K}$ is of full row rank.[4]

Under this assumption, we can get a closed-form solution for the expectation matrices $\widetilde{\boldsymbol{S}}_K$ and $\boldsymbol{S}_i - \boldsymbol{S}_K$ from equation (11). Note that expectation matrix $\widetilde{\boldsymbol{S}}_K$ describes subjective expectations for choice K in a state $x \in \{1, 2, \cdots, J-1\}$ relative to state J. Without further restrictions, we are unable to *fully* identify subjective expectations conditional on choice K from the identified matrix $\widetilde{\boldsymbol{S}}_K$. To achieve full identification, we impose the

[4] Assumption 5A (b) guarantees that $\Delta\boldsymbol{\xi}_{i,K}$ has an inverse or a right inverse, when $T = 2J$ or $T > 2J$, respectively.

following normalization to pin down the expectation vector for the reference state J, i.e., $\boldsymbol{S}_K(J)$.

Assumption 6A *There exists a state x, in $\{1, 2, \cdots, J\}$ under which agents' subjective expectations about the state transition are known for action K.*

The restriction of known subjective expectations imposed in assumption 6A is only required to hold for a certain state and action. Without loss of generality, we assume $S_K(J)$ is known. A sufficient condition for known $S_K(J)$ is that agents hold rational expectations for action K in state J and the econometrician knows this fact. Subjective expectations need to be partially known is due to nature of DDC models. Specifically, the CCP for any action does not depend on its choice-specific value fully, instead it depends on choice-specific values of all actions relative to that of a reference action, say, K, and these relative choice-specific values further rely on expectations for i relative to K. This implies that a normalization is necessary to fully identify subjective expectations for all actions and states. A possibility of relaxing the normalization is to explore other restrictions provided outside the model, e.g., the relationship between subjective expectations and objective transitions.

We consider an example where a single woman makes dynamic labor participation decisions (work or not work). The state variable is her income (high, medium, and low), and she has to form expectation regarding income in the future given current income and working status, i.e., the income transition. The normalization in this context is reasonable in the sense that for women with low income of not working, they have a very good understanding of their income in the future because in such a scenario the only income source is some social welfare programs she qualifies. This indicates that the subjective expectation is the same as its objective counterpart conditional on low income and non-working.

Under Assumption 6A, we can identify the subjective expectations matrix associated with action K, $\boldsymbol{S}_K$. Consequently, expectation matrix $\boldsymbol{S}_i$ is identified from $\boldsymbol{S}_i - \boldsymbol{S}_K$. The identification results of the *ex-ante* value difference $\Delta \boldsymbol{V}_t$ and expectation matrices $\boldsymbol{S}_i$ and $\boldsymbol{S}_K$ can be used to identify expectation matrices for actions $i' \neq i$ and $i' \neq K$ using equation (6) of action i', which does not require any further assumptions. We summarize our identification results as follows.

Theorem 1 *Suppose that Assumptions 1–4, 5A, and 6A hold. The subjective expectations $s(x'|x, a)$ for $x, x' \in \{1, 2, ..., J\}$ and $a \in \{1, 2, ..., K\}$ are identified as a closed-form function of the CCPs, $p_t(a|x)$, for $t = 1, 2, \cdots, T$, $T \geq 2J$.*

The identification results in Theorem 1 require at least $2J$ consecutive periods of observations or $2J - 2$ cells of three consecutive periods. This requirement could be restrictive in some empirical applications, especially when the state space is large. We next present an alternative strategy where $J + 1$ periods of data are sufficient for identification.

Assumption 5B *(a). The number of periods observed is not smaller than $J + 1$, i.e., $T \geq J + 1$. (b). The matrix $(\Delta \boldsymbol{\xi}^1_{i,K})' \otimes (\beta \widetilde{\boldsymbol{S}}_K) - (\Delta \boldsymbol{\xi}^2_{i,K})' \otimes I$ is of full column rank.*

Assumption 6B *There exists an action $a = K$ under which agents' subjective expectations about the law of motion for choice K are known.*

The matrix $(\Delta\boldsymbol{\xi}^1_{i,K})'\otimes(\beta\widetilde{\boldsymbol{S}}_K)-(\Delta\boldsymbol{\xi}^2_{i,K})'\otimes I$ is of size $(T-2)\cdot(J-1)$ by $(J-1)\cdot(J-1)$, where I is a $J-1$ by $J-1$ identity matrix and $\otimes$ denotes the Kronecker product. Assumption 5B is similar to the full rank condition 5A: both require sufficient variation in CCPs over time, and both assumptions are empirically testable. Under Assumption 6B, there are $J-1$ unknown parameters in the linear system (11). Following the discussion of Assumption 5A, we need $T-2 \geq J-1$, i.e., $T \geq J+1$ to solve the linear system if $\boldsymbol{S}_K$ is known. Once $\beta\widetilde{\boldsymbol{S}}_K$ is known, Assumption 5B(b) imposes restrictions on the observed log ratios of CCPs, $\Delta\boldsymbol{\xi}^1_{i,K}$, and $\Delta\boldsymbol{\xi}^2_{i,K}$.

Assumption 6B requires that the expectation matrix associated with action K is known; i.e, $\boldsymbol{S}_K$ is known, which is stronger than the normalization assumption in Assumption 6A. Suppose we consider a dynamic investment problem, where an agent chooses to invest in stock market or save in a saving account based on her subjective expectations on the transition of the state variable, wealth. The accumulated wealth in future conditional on current wealth and saving in the bank should be very straightforward to predict given there is minimum risk. That is, agents have rational expectation if they put their money in a bank. In contrast, the accumulated future wealth is more difficult to predict because of market volatility if the agent chooses to invest.

We state the identification result under Assumptions 5B and 6B in the following theorem; we omit the proof since it is similar to the proof of Theorem 1.

Theorem 2 *Suppose that Assumptions 1–4, 5B, and 6B hold. The subjective expectations $s(x'|x,a)$ for $x, x' \in \{1,2,...,J\}$ and $a \in \{1,2,...,K-1\}$ are identified as a closed-form function of the CCPs, $p_t(a|x)$, $t = 1,2,...,T$, $T \geq J+1$.*

Theorems 1-2 demonstrate that it is indeed difficult to disentangle agent expectations from preferences by observing only their choices over time. Even if expectations and preferences are time-invariant, identification still requires a sufficiently large number of periods and some rank conditions. Moreover, a full set of subjective expectations cannot be point identified, i.e., normalization is required. Theorems 1-2 illustrate the trade-off between normalization and the requirement on the data.[5] In particular, if we have more information regarding subjective expectations (a stronger normalization assumption), identification would require less variation (shorter time periods of data and a different rank condition). The benefit of imposing a stronger normalization assumption is that it significantly reduces the number of observed periods necessary for identification. This is especially helpful in the empirical applications where the state variable takes a large number of values.

In Theorems 1-2, we neither impose restrictions on the continuation value in the terminal period nor require the observation of data in the terminal period. Naturally, if the data for the terminal period are available and we are willing to assume that the

[5] Even the subjective expectations in Theorems 1-2 are only partially identified due to normalization, they would be important in testing whether the assumption of rational expectations is valid.

continuation value at the terminal period is zero, we can achieve identification of the subjective expectations using fewer periods of data. This is because we can identify the flow utility using the CCP at the terminal period under the assumption that there is no continuation value at the terminal period. With identification of the flow utility, there is no need to control for the flow utility's impact on the log ratio of CCPs by taking a difference. Therefore, identification of the flow utility decreases the required number of periods by one. We leave the details of the identification to the Appendix.

Remark 1. Given that the discount factor and the distribution of the unobserved state variable are known and subjective expectations are identified, we can non-parametrically identify the utility function relative to action K, i.e., $u_a(x) - u_K(x)$, following Magnac and Thesmar (2002). Furthermore, the discount factor can be identified if we are willing to impose a stronger normalization condition (6B) and rank condition (Assumptions 5A). The intuition is that a stronger normalization allows extra information so that discount factor β is identified by using time as an exclusion restriction. This is because the discount factor affects the log ratio of CCPs in a way that is similar to the expectation difference $S_i - S_K$ in the key equation (4). Specifically, we can identify $\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}$ and $\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}$ separately from equation (11). Under Assumption 6B, $\widetilde{\boldsymbol{S}}_K$ is known, which allows us to identify $\boldsymbol{S}_i - \boldsymbol{S}_K$. Consequently, both $\boldsymbol{S}_i$ and β are identified.

4 Identification with additional state variables

The identification strategy in Section 3 uses time as an exclusion restriction for both belief and flow utility because it shifts CCPs without affecting preference or expectations in the finite horizon framework. In this section, we extend the identification argument to the case where an additional state variable, which evolves independently from the state variables admits subjective beliefs, i.e., exclusion restriction to subjective beliefs. Specifically, suppose there exists an additional state variable w, which is also discrete, i.e., $w \in \{1, 2, \cdots, M\}$. We impose the following assumptions on the objective transition of w and agents' subjective expectations on this objective counterpart.

Assumption 7 (expectation exclusion) *(a) The observed state variables x and w evolve independently, and the transition of w is exogenous, i.e,*

$$f(x', w'|x, w, a) = f(x'|x, a)f(w'|w).$$

(b) Agents believe that state variables x and w evolve independently and have rational expectations on the evolution of w.

$$s(x', w'|x, w, a) = s(x'|x, a)s(w'|w) = s(x'|x, a)f(w'|w). \tag{12}$$

In addition to independent transitions, Assumption 7(a) restricts the state variable w to be at the "macro level" such that agents' actions have no impact its transition. This restriction can be relaxed for in the infinite horizon framework. Assumption 7(b) imposes two restrictions on agents' subjective expectations. First, agents correctly predict that the two state variables are independently evolving. Second, agents have rational expectations on the transition of state variable w. Note that the independence of the transitions for state variables x and w in Assumption 7(a) is often assumed in the literature (e.g., see the applications of DDC models reviewed in Aguirregabiria and Mira (2010)). The assumption of knowing transition of w can be rationalized by the fact that agents often have better understanding of the transition of some variables over that of others. Especially, under the assumption that w is a macro-level variable and its transition does not depend on agent choice, agents' subjective expectations about its transition can be accurate (see e.g., Manski (2004)).

In the literature of DDC models, an exclusive variable is often used for identification. For example, in Abbring and Daljord (2018), an exclusion restriction is used to identify the discounting factor. Oftentimes, such an exclusive variable, say Z, is assumed to have a pair of states, denoted as z_1 and z_2, such that the flow utility does not change across the two values, i.e., $u(z_1, a) = u(z_2, a)$. Nevertheless, the exclusion restriction introduced in our setting is more restrictive than the traditional exclusion restriction due to the following reasons. First, the exclusive variable w evolves independently from x – exclusion for expectations. This is necessary because our objective is to identify subjective expectations on x, and if w also shifts expectations, the number of unknowns increases at the same order as the variation so that there is no benefit of adding this additional state variable. Second, in the finite horizon framework, this additional state variable does not need to be an exclusion restriction to flow utility because time still provides exclusion restrictions for both subjective expectations and the flow utility. As a result, existence of w provides additional variation to identify subjective expectations. In the infinite horizon case, however, time cannot serve as exclusion restriction due to stationarity. The additional state variable needs to be exclusive for both subjective expectations and the flow utility. In the reminder of this section, we will discuss identification of DDC models in finite and infinite horizon frameworks separately.

4.1 Identification of finite horizon models

In the finite horizon framework, both time and the additional exclusive variable, w, can provide exclusion restrictions for identification under Assumption 7. Combining the two exclusive variables increases variation in the log ratio of CCPs which makes the identification of expectations easier than only relying on time. We first rewrite the log ratio of CCPs in period t with the additional state variable.

$$
\begin{aligned}
\xi_{t,i,K}(x,w) \;\equiv\;& \big[u_i(x,w) + \beta \sum_{x',w'} V_{t+1}(x',w') f(w'|w) s(x'|x,i)\big] \\
& - \big[u_K(x,w) + \beta \sum_{x',w'} V_{t+1}(x',w') f(w'|w) s(x'|x,K)\big]. \\
\equiv\;& u_i(x,w) - u_K(x,w) + \beta(S_i(x) - S_K(x)) \boldsymbol{V}_{t+1} F(w), \qquad (13)
\end{aligned}
$$

where $F(w) \equiv [f(w'=1|w), ..., f(w'=M|w)]'$ is a $M \times 1$ vector of distributions for future state w' conditional on w, and $\boldsymbol{V}_t$ is the $(J-1) \times M$ *ex-ante* value matrix with the $j$the column being the vector of relative value using $x = J$ as the reference state for $w = k$, i.e., $\boldsymbol{V}_t \equiv [V_t(x=j, w=k) - V_t(x=J, w=k)]_{j,k}$, for $j = 1, ..., J-1$ and $k = 1, ..., M$.

Following an identification argument similar to that in Section 3, we also assume that preferences are time-invariant to rule out the impact of flow utility on the log ratio of CCPs over time. By slight abuse of notation, we use $\Delta\xi_{t,i,K}(x,w)$ to represent the first difference of CCP log ratios when we have the additional state variable w. That is,

$$\begin{aligned} \Delta\xi_{t,i,K}(x,w) &\equiv \xi_{t,i,K}(x,w) - \xi_{t-1,i,K}(x,w) \\ &\equiv \beta[S_i(x) - S_K(x)]\Delta \boldsymbol{V}_{t+1} F(w), \end{aligned} \tag{14}$$

where $\Delta \boldsymbol{V}_{t+1} \equiv \boldsymbol{V}_{t+1} - \boldsymbol{V}_t$ is the first difference of $\boldsymbol{V}_{t+1}$, so it is a $(J-1) \times M$ matrix. This equation indicates that the over-time change in CCP log ratio conditional on states x and w is determined by the expectation difference for state x, the difference of the *ex-ante* value functions, and the law of motion for the extra state variable w. A comparison between equations (5) and (14) demonstrates that the effect of the additional state variable w on the log ratio of CCPs can be separated from agents' subjective expectations on x with the assumption that they evolve independently.

We collect the moment conditions above with $J-1$ values of x and all values of w to generate the the following matrix representation.

$$\Delta\boldsymbol{\xi}_{t,i,K} \equiv \beta\big[\boldsymbol{S}_i - \boldsymbol{S}_K\big]\Delta \boldsymbol{V}_{t+1} \boldsymbol{F}_w, \tag{15}$$

where the CCP log ratio matrix $\Delta\boldsymbol{\xi}_{t,i,K}$ is defined similar to the CCP log ratio vector in the previous section with the jth column being the vector of the log ratio of CCPs for $w = k$, i.e, $\Delta\boldsymbol{\xi}_{t,i,K} \equiv [\Delta\xi_{t,i,K}(x=j, w=k)]_{j,k}$, where $j = 1, ..., J-1$ and $k = 1, ..., M$, and matrix $\boldsymbol{F}_w$ captures the overall transition matrix of w, i.e., $\boldsymbol{F}_w \equiv [F(w=1), ..., F(w=M)]$. The equation above is the matrix version of equation (6).

Following the identification argument in the previous section, we also explore the recursive relation of the value function over time to provide extra restrictions on expectations. By imposing the full rank condition on $\boldsymbol{S}_i - \boldsymbol{S}_K$ (Assumption 4), combined with the recursive property of the *ex-ante* value function, we can obtain the main identification equation as follows.

$$\Big[\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}, \quad -\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Big]\Delta\boldsymbol{\xi}_{i,K} = \Delta \log \boldsymbol{p}_K \boldsymbol{F}_w, \tag{16}$$

where $\Delta \log \boldsymbol{p}_K$ collects all first-difference CCPs over time and across states and is defined as in the previous section; $\Delta\boldsymbol{\xi}_{i,K}$ collects all first-differences of the log ratio of CCPs over time and across states and is defined similar to that in the previous section with adjustment using the transition of state w, i.e., $\Delta\boldsymbol{\xi}_{i,K} \equiv \begin{bmatrix} \Delta\boldsymbol{\xi}_{3,i,K}\boldsymbol{F}_w & \Delta\boldsymbol{\xi}_{4,i,K}\boldsymbol{F}_w & ... & \Delta\boldsymbol{\xi}_{T,i,K}\boldsymbol{F}_w \\ \Delta\boldsymbol{\xi}_{2,i,K} & \Delta\boldsymbol{\xi}_{3,i,K} & ... & \Delta\boldsymbol{\xi}_{T-1,i,K} \end{bmatrix}$. The matrix of the first-differences of the log ratio of CCPs, $\Delta\boldsymbol{\xi}_{i,K}$, is of dimension $(2J-2) \times \big(M \cdot (T-2)\big)$.

Equation (16), similar to equation (11) with time variation only, is the key identification equation where variations of CCPs are from both time and state variable w. Intuitively, the existence of the additional state variable w greatly reduces the time periods required for identification. However, the variation from the extra state variable has to be sufficient, meaning that a rank condition needs to be satisfied. Specifically, we impose an assumption similar to Assumptions 5A and 5B as follows.

Assumption 5C *(a). The number of periods observed, T, and the cardinality of the additional state variable, M, satisfies the following inequality, i.e., $M(T-2) \geq 2J-2$. (b). The matrix $\Delta\boldsymbol{\xi}_{i,K}$ is of full row rank.*

Assumption 5C(a) requires that the total number of time periods T is no fewer than $2+(2J-2)/M$, which can be much smaller than J if M is large. A comparison between (16) and (11) demonstrates that the existence of the additional state variable augments each restriction to M ones for any given state x while the number of unknown parameters is still $2(J-1)$. Solving the linear system (16) requires the number of restrictions $M(T-2) \geq 2J-2$, and the matrix $\Delta\boldsymbol{\xi}_{i,K}$ is of full row rank, which is empirically testable.

Under the assumption above, we follow the identification strategy in Section 3 to identify all the subjective expectations with some normalization. Let us use $\lceil y \rceil$ to denote the minimum integer greater than or equal to y. The results are summarized in the following theorem.

Theorem 3 *Suppose that Assumptions 1–4, 5C, 6B, and 7 hold. The subjective expectations $s(x'|x,a)$ for $x, x' \in \{1,2,...,J\}$ and $a \in \{1,2,...,K-1\}$ are identified as a closed-form function of the CCPs, $p_t(a|x)$, $p_{t-1}(a|x)$, and $p_{t-2}(a|x)$, for $t = 1,...,T$, $T \geq \lceil \frac{2J-2}{M} \rceil + 2$.*

The required number of periods of data in Theorem 3 reveals complementarity of the additional state variable and time in terms of providing variation for identification. If M is of order J, the number of periods required for identification does not depend on the cardinality of the support of x (J). For example, in the case of $M = J$, four periods of data are sufficient for identification; if $M \geq 2J-2$, three periods are sufficient for identification. The fewer time periods required for identification in Theorem 3 is of great importance in empirical applications where the state variable may take a large number of possible values.

4.2 Identification of infinite horizon models

The identification strategy in Section 4.1 cannot be directly extended to the infinite horizon framework because CCPs are time-invariant due to stationarity. Therefore, we consider possible variations of CCPs across states of w and show that the model is identified if w is exclusion restriction for both subjective expectations (Assumption 7) and flow utility.

Assumption 8 (preference exclusion) *(a) The utility function for choice $a = K$ is normalized, $u_K(x, w) = 0$ for any x and w. (b) There exists at least $J - 1$ pairs of values of the additional state variable w, such that $\forall i \neq K$ and any $x \in \mathcal{X}$,*

$$u_i(x, w_1^j) = u_i(x, w_2^j), \quad j = 1, ..., J - 1, \tag{17}$$

where $w^j \equiv \{w_1^j, w_2^j\}$ is the j-th pair of $\{w_1, w_2\}$.

Assumption 8(a), the normalization assumption, is widely imposed in the literature of dynamic discrete choice models and discrete games due to the fact that we can only identify the relative flow utility (Rust (1994)). The exclusive feature of the additional state variable imposed in Assumption 8(b) has been exploited in the existing literature (see e.g., Abbring and Daljord (2018)). Note that our identification requires that there exists $J - 1$ such pairs, which indicates that state variable w takes at least J values, i.e., $M \geq J$. Moreover, the $J - 1$ pairs can be different across actions.

It is necessary to impose Assumption 8 on flow utility for identifying infinite horizon models because we need to control the impact of flow utility on CCPs through the additional state variable w. In finite horizon models, this is realized by using the assumption of stationary flow utility.

Because of stationarity in the infinite horizon framework, we drop the index of time in the notation. The log ratio of CCPs can be rewritten as

$$\xi_{i,K}(x, w) \equiv u_i(x, w) - u_K(x, w) + \beta(S_i(x) - S_K(x))\boldsymbol{V}F(w), \tag{18}$$

where $S_i(x), S_K(x), \boldsymbol{V}$, and $F(w)$ are defined analogous to that in equation (13).

Unlike in the case of finite horizon, we cannot rely on variations in the log ratio of CCPs over time to identify expectations. Instead, we explore variations in the log ratio of CCPs across the special pairs w^j. Under Assumption 8, $u_i(x, w) - u_K(x, w)$ is invariant for any such pairs; therefore, we have

$$\begin{aligned} \Delta\xi_{i,K}(x, w^j) &\equiv \xi_{i,K}(x, w_1^j) - \xi_{i,K}(x, w_2^j) \\ &= \beta(S_i(x) - S_K(x))\boldsymbol{V}[F(w_1^j) - F(w_2^j)] \\ &\equiv \beta(S_i(x) - S_K(x))\boldsymbol{V}\Delta F(w^j), \end{aligned} \tag{19}$$

where $\Delta F(w^j) \equiv F(w_1^j) - F(w_2^j)$ of dimension $M \times 1$, captures the difference in evolution of the future value of the exclusion restriction pair w^j.

To identify expectations $S_i(x) - S_K(x)$ from equation (19), we need to recover the unknown future values contained in $\boldsymbol{V}$, which depend on both preferences and expectations. We show in the following lemma that $\boldsymbol{V}$ is identified if the subjective expectation associated with the reference action K is known, i.e., Assumption 6B holds, and the preference associated with this reference action is normalized to zero, which is stated below.

Lemma 1 *Under Assumptions 1-3, 6B, 7, and 8(a), the value function $V(x, w)$ is identified.*

We sketch the proof of Lemma 1 here and leave the details to the Appendix. Under Assumption 8(a), we can express the *ex-ante* value function $V(x, w)$ as

$$\begin{aligned} V(x,w) &= -\log p_K(x,w) + v_K(x,w) \\ &= -\log p_K(x,w) + u_K(x,w) + \beta \sum\nolimits_{x',w'} V(x',w') f(w'|w) s(x'|x,K) \\ &= -\log p_K(x,w) + \beta \sum\nolimits_{x',w'} V(x',w') f(w'|w) s(x'|x,K). \end{aligned} \tag{20}$$

Intuitively, we could keep iterating this equation to approximate the *ex-ante* value $V(x, w)$ by collecting all the terms: $-\log p_K(x, w)$, which is observed, and S_K and $f(w'|w)$, which are assumed to be known under Assumptions 6B and 7, respectively. Technically, the expression above provides a fixed point equation for $V(x, w)$ and it can be directly solved as a closed-form function of $-\log p_K(x, w)$, S_K, and $f(w'|w)$.

Note that both preferences and expectations need to be normalized for the same reference action K in order to identify the value function. Such requirements are due to the fact that, in infinite horizon models, the value function is the fixed point to a contraction mapping involving preferences and expectations for any action. Normalizing preferences and expectations for any action allows us to recover the value function, which has a non-linear relationship with expectations. By contrast, in finite horizon models, identification does not require us to recover the value functions. This is because (1) the CCPs in any period of time reveal information on the *ex-ante* value function in this period, thus the first-difference of the CCPs provide information about the first-difference of the value functions; and (2) the value functions admit a recursive relationship over time. The additional recursive relationship enables us to control the impact of the value function on CCPs in a finite horizon framework.

Once the *ex-ante* value function is identified, equation (19) provides an equation with the subjective expectation vector $S_i(x)$ as the only unknown. We show that $J - 1$ equations as in (19) provide sufficient variation for us to achieve identification of $S_i(x)$. To better understand the conditions required for identification and to obtain a closed-form expression for subjective expectations, we stack equation (19) for any x and all pairs of w into the following matrix equation.

$$\boldsymbol{\Delta\xi}_{i,K}(x) \;=\; \beta(S_i(x) - S_K(x))\boldsymbol{V}\Delta\boldsymbol{F}, \tag{21}$$

where $\boldsymbol{\Delta\xi}_{i,K}(x) \equiv [\Delta\xi_{i,K}(x, w^1), ..., \Delta\xi_{i,K}(x, w^{J-1})]$ is of dimension $1 \times (J - 1)$ and $\Delta\boldsymbol{F}$, of dimension $M \times (J - 1)$, both defined analogously.

Equation (21) provides restrictions with subjective expectations $S_i(x)$ as the only unknown. Therefore, $S_i(x)$ can be identified if there is sufficient variation in (21), which is guaranteed by the following assumption.

Assumption 5D *The* $(J - 1) \times (J - 1)$ *matrix* $\boldsymbol{V}\Delta\boldsymbol{F}$ *is of full rank.*

This assumption is empirically testable because we can identify the value function from CCPs directly. Assumption 5D is equivalent to assuming that value matrix $\boldsymbol{V}$ and transition matrix $\Delta\boldsymbol{F}$ for the state variable w are full row rank and full column rank, respectively. This requires that both the $J-1$ rows, $V(x=j,w=k)-V(x=J,w=k)$, for $x=1,2,\cdots,J-1,k=1,2,\cdots,M$ and the $J-1$ transition vector, $F(w_1^j)-F(w_2^j)$, for $j=1,2,\cdots,J-1$ are linearly independent. Under this assumption, we can identify the expectations associated with action i as

$$S_i(x)=S_K(x)+\beta^{-1}\boldsymbol{\Delta\xi}_{i,K}(x)\big[\boldsymbol{V}\Delta\boldsymbol{F}\big]^{-1}. \tag{22}$$

Note that the result above applies to all actions, so we can identify the expectations associated with other actions analogously. We summarize the results of identification for the infinite horizon case as follows.

Theorem 4 *Suppose that Assumptions 1-3, 5D, 6B, 7- 8 are satisfied. The subjective expectations $s(x'|x,a)$ for $x,x'\in\{1,2,...,J\}$ and $a\in\{1,2,...,K-1\}$ are identified as a closed-form function of the CCPs, $p(a|x,w)$, the objective state transition $f(w'|w)$, and the expectation of $S_K(x)$.*

Recall that Assumption 7 requires the state transition of w to be independent of agents' action a. This restriction can be relaxed and Theorem 4 still holds if agent actions do affect the transition of w. The derivation is similar to Theorem 4 and we omit it in the paper.

Remark 2. The identification argument of Theorem 4 can be readily applied to the finite horizon model if $J-1$ pairs of w exist as in Assumption 8(a), and at least the last two periods of data are observed. Specifically, in finite horizon models, we can follow the identification procedure of Theorem 4 with appropriate adjustments to the assumptions for identifying agent expectations in two steps. First, we recover *ex-ante* value functions V_t for $t=T-1$ and T using the recursive relationship under Assumption 8(b) (preference normalization). Second, we use the connection between CCPs, expectations, and the identified value function for period $T-1$, which is similar to equation (22), to identify the expectations. In summary, we only need the last two periods of data to identify agent expectations, which improves the result in Theorem 3 in terms of data requirements.

Remark 3. In some empirical applications, state variables can be decomposed into states that evolve independently of agents' actions (e.g., wages) and states that are deterministic functions of past actions and/or states (e.g., experience, age). In those applications, it is reasonable to assume that the deterministic transition is known to agents. We show in the Appendix that if we make these alternative assumptions on state variables instead of Assumption 7, we can also follow the procedure in Theorem 3 to identify the (action-independent) subjective expectations in finite horizon models. We summarize the result in Theorem A.1 in the Appendix. Particularly, we show in Theorem A.1 that (1) normalization of expectations such as Assumptions 6A and 6B are not necessary for identification, and (2) the number of periods of data required for identification is $\lceil\frac{J-1}{M-1}\rceil+2$. These results are improvements over Theorem 3, which is based on Assumption 7. The improvements are mainly due to the exogenous transition of the state variable x.

First, when the expectations evolve endogenously, the useful information for identification is from the relative CCPs to a reference choice, say K. Thus, normalization is necessary. In the exogenous case, however, the expectations do not depend on any actions and a reference action is no longer necessary. Second, exclusion restrictions in both Theorems 3 and A.1 are from the state variable w and time, and this greatly reduces data requirement. Nevertheless, exogenous transition of the state variable in Theorem A.1 further decreases the number of periods of data from $\lceil \frac{2J-2}{M} \rceil + 2$ to $\lceil \frac{J-1}{M-1} \rceil + 2$ (note that $\frac{2J-2}{M} \geq \frac{J-1}{M-1}$ for any $M \geq 2$). Specifically, if $M \geq J$, we can identify the expectations using only three consecutive periods of data.

The result of the identification above does not apply to infinite horizon models. Under the assumptions of the exogenous transition of x, variation in CCP across actions provide less identifying power for subjective expectations than in the case of endogenous transition of x, and this prevents separation of the value function from expectations. On the other hand, the value function in the infinite horizon is a fixed point solution to a contraction mapping that depends on expectations and flow utility (normalized to be zero). As a result, the value function has a nonlinear relationship with expectations, making it impossible to disentangle the value function from expectations. In Theorem 4, the value function can be identified with extra normalization to the expectations associated with the reference action, which is infeasible here when expectations are exogenous. To summarize, the value function and expectations cannot be separately identified because both are dependent on state variable x but not action a.

5 An Extension: Heterogeneous expectations

Agents may display heterogeneous preferences and/or expectations about the transition of the same state variable. We show in this section that a DDC model with such heterogeneity can also be identified using the results in previous sections. Suppose agents can be classified into $H \geq 2$ types, and H is known to the econometrician.[6] Let $\tau \in \{1, 2, ..., H\}$ denote the unobserved type (heterogeneity) such that all agents of the same type have the same subjective expectations and preferences, denoted as $s(x'|x, a, \tau)$ and $u_a(x, \tau)$, respectively. Similarly, the CCP for agents of type τ in period t is $p_t(a|x, \tau)$. An agent's type is assumed to remain unchanged over time. We use an identification methodology developed in the measurement error literature i.e., Hu (2008), to show that the observed joint distribution of state variables and agent actions uniquely determines the type-specific CCP $p_t(a|x, \tau)$ for all $\tau \in \{1, 2, \cdots, H\}$. We then apply the results of identification in Sections 3-4 to identify the heterogeneous expectations $s(x'|x, a, \tau)$ and utility functions $u_a(x, \tau)$ associated with type τ. We focus our discussion on the finite horizon case, and the procedure is readily applicable to the infinite horizon case.

[6] The number of types H may be inferred from the data, see e.g., Kasahara and Shimotsu (2009). For ease of exposition, we assume H to be known.

Intuitively, the observed CCP $p_t(a|x)$ is a finite mixture of the H components $p_t(a|x,\tau)$.

$$p_t(a|x) = \sum\nolimits_{\tau=1}^{H} p_t(a|x,\tau)q(\tau|x), \tag{23}$$

where $q(\tau|x)$ is the conditional probability of type τ given state x. Hu (2008) shows that the unknowns on the right-hand-side of the equation can be nonparametrically identified if there exist two measurements of the latent type variable τ and a binary variable that is correlated with τ. In our DDC model, we use actions as the measurements of τ and the state variable as the one correlated with τ. In the rest of this section, we present all the assumptions required for identifying the CCPs $p_t(a|x,\tau)$ and leave the proof of identification for the Appendix.

Assumption 9 (markov property) *Given any type τ, $\{a_t, x_t\}$ follows a first-order Markov process.*

The first-order Markov property of action and state variables is widely assumed in the literature on DDC models. A sufficient condition for this assumption is that the state variable evolves according to a first-order Markov process, and an agent's decision only depends on the current state. Under this assumption, the observed joint distribution of state variable and action is then associated with the unobserved types as follows:

$$\begin{aligned}
& \Pr(a_{t+l},...,a_{t+1},x_{t+1},a_t,x_t,a_{t-1},...,a_{t-l}) \\
= & \sum\nolimits_{\tau=1}^{H} \Pr(a_{t+l},...,a_{t+1}|x_{t+1},\tau)\Pr(x_{t+1},a_t|x_t,\tau)\Pr(\tau,x_t,a_{t-1},...,a_{t-l}) \\
= & \sum\nolimits_{\tau=1}^{H} \Pr(a_{t+l},...,a_{t+1}|x_{t+1},\tau)\Pr(a_t|x_{t+1},x_t,\tau)\Pr(x_{t+1}|x_t,\tau)\Pr(\tau,x_t,a_{t-1},...,a_{t-l}).
\end{aligned}$$

The joint distribution above is expressed as a misclassification model, with actions $\{a_{t+l},...,a_{t+1}\}$ and $\{a_{t-1},...,a_{t-l}\}$ being two measurements for the unobserved types (Hu (2019)). We require the integer l to satisfy $H \le K^l$ such that the cardinality of the support for $\{a_{t+l},...,a_{t+1}\}$ and $\{a_{t-1},...,a_{t-l}\}$ is not smaller than that of the unobserved type τ. Otherwise, the latent variable takes more values than its measurements and the model can not be identified.

To apply the identification strategy of eigenvalue-eigenvector decomposition from the measurement error literature, e.g., Hu (2008), we reduce the cardinality of the measurements to be the same as that of the latent unobserved types through a known function $h(\cdot)$. Specifically, function $h(\cdot)$ maps the support of $(a_{t+l},...,a_{t+1})$, K^l, to that of τ, $\{1,2,...,H\}$. We define the two measurements for the latent type as,

$$\begin{aligned}
a_{t+} &\equiv h(a_{t+l},...,a_{t+1}), \\
a_{t-} &\equiv h(a_{t-1},...,a_{t-l}).
\end{aligned} \tag{24}$$

We exemplify function $h(\cdot)$ as follows. Suppose we consider a dynamic investment problem where agents choose whether (or not) to invest in a stock ($a_t \in \{1,0\}$, $a_t = 1$ and $a = 0$ stand for "invest" and "do not invest", respectively). An agent's decision depends both on a discrete state variable x_t that describes her financial status and her subjective expectations about the return of the stock. For illustrative purposes, we assume that agents have homogenous preferences and heterogeneous expectations. The unobserved type captures the degree of accuracy of subjective expectations, i.e., how close they are to the *ex-post* distribution of returns, and it takes three values: "more accurate," "accurate," and "less accurate," denoted as $\tau = 1, 2$, and 3, respectively. In this setting, $H = 3$, and it is sufficient to choose $l = 2$ such that $H = 3 < K^l = 4$. The function $h(\cdot)$ maps the support of actions in two consecutive periods, i.e., $\{(0,0), (0,1), (1,0), (1,1)\}$, to the space of the unobserved types, i.e., $\{1, 2, 3\}$. One possible choice of $h(\cdot)$ is

$$a_{t+} = \begin{cases} 1, \text{ if } (a_{t+1}, a_{t+2}) = (0,0), \text{invest in neither periods} \\ 2, \text{ if } (a_{t+1}, a_{t+2}) = (0,1) \text{ or } (a_{t+1}, a_{t+2}) = (1,0), \text{invest in one of the two periods} \\ 3, \text{ if } (a_{t+1}, a_{t+2}) = (1,1), \text{invest in both periods.} \end{cases}$$

The other measurement a_{t-} can be similarly defined.[7]

For a given pair (x_t, x_{t+1}), we define a matrix to describe the joint distribution of a_{t+1} and a_{t-}.

$$M_{a_{t+},x_{t+1},x_t,a_{t-}} = \left[f(a_{t+} = i, x_{t+1}, x_t, a_{t-} = j)\right]_{i,j}.$$

Our identification requires the H by H matrix $M_{a_{t+},x_{t+1},x_t,a_{t-}}$ to be of full rank.

Assumption 10 *For all $(x_{t+1}, x_t) \in \mathcal{X} \times \mathcal{X}$, the rank of the matrix $M_{a_{t+},x_{t+1},x_t,a_{t-}}$ is H.*

In the investment example above, this assumption requires that the three columns of the 3 by 3 matrix $M_{a_{t+},x_{t+1},x_t,a_{t-}}$ describing the joint distribution of a_{t+}, a_{t-}, x_t, and x_{t+1} are linearly independent for any given x_t and x_{t+1}. Assumption 10 is imposed on a matrix that is directly estimable from the data, and therefore the assumption is empirically testable. Moreover, because both a_{t+} and a_{t-} depend on mapping $h(\cdot)$, which is not unique, we may choose $h(\cdot)$ to ensure that the assumption holds.

Following Hu (2008), our identification strategy involves an eigenvalue-eigenvector decomposition of an estimable matrix, and such a decomposition must be unique. The uniqueness is satisfied by distinctive eigenvalues of the decomposition.

Assumption 11 (distinctive eigenvalues) *For any given $(x_{t+1}, x_t) \in \mathcal{X} \times \mathcal{X}$, there exists a choice $k \in \mathcal{A}$ such that $\Pr(a_t = k|x_{t+1}, x_t, \tau)$ differs for any two different values of $\tau \in \{1, 2, \cdots, H\}$.*

To investigate the restrictions Assumption 11 imposes to our model, we write the probability $\Pr(a_t = k|x_{t+1}, x_t, \tau)$ as follows.

$$\Pr(a_t = k|x_{t+1}, x_t, \tau) = \frac{\Pr(x_{t+1}|x_t, a_t = k)\Pr(a_t = k|x_t, \tau)}{\sum_{a_t \in \mathcal{A}} \Pr(x_{t+1}|x_t, a_t)\Pr(a_t|x_t, \tau)}.$$

[7] It is clear from the example that the choice of $h(\cdot)$ is not unique. As we show in the Appendix, given that the assumptions of identification hold, the choice of $h(\cdot)$ does not affect our estimation results.

The equality holds because the transition of x, $\Pr(x_{t+1}|x_t, a_t)$, does not depend on agent type τ. An important implication of the equation above is that if agent actions are binary, then a sufficient condition of Assumption 11 is that agent CCP $\Pr(a_t = k|x_t, \tau)$ differs for any two types.

In the investment example discussed above, this sufficient condition requires that, given financial status in period t, agents with different types would choose to invest (or not) with different probabilities. This requirement indicates that expectations of different types need to differ sufficiently such that, given the state variable, their CCPs are distinct. It is worth noting that the assumption is only required to hold for an action k rather than for all actions.

Assumption 12 (monotonicity) *For any given $x_{t+1} \in \mathcal{X}$, there exists a known $m \in \{1, 2, \cdots, K^l\}$ such that $\Pr(a_{t+} = m|x_{t+1}, \tau)$ is strictly monotonic in τ.*

Let us interpret the restriction this assumption imposes to the model using the investment example above. Suppose $a_{t+} = (1, 1)$ satisfies the assumption such that $\Pr(a_{t+} = (1, 1)|x_{t+1}, \tau)$ is strictly decreasing in τ, then it states that the more accurate the subjective expectations, the higher the probability with which agents choose to invest in both periods of $t + 1$ and $t + 2$. Recall that a_{t+} is defined as mapping $h(a_{t+l}, ..., a_{t+1})$ and the choice of $h(\cdot)$ is not unique. This gives us some flexibility to choose $h(\cdot)$ such that Assumption 12 holds. The objective $\Pr(a_{t+} = m|x_{t+1}, \tau)$ is different from a CCP, thus Assumption 11 is either sufficient nor necessary for Assumption 12. In empirical applications, the sufficient conditions of these two assumptions are often model specific, e.g., see a similar assumption in An, Hu, and Shum (2010).

We summarize the identification result in the following theorem.

Theorem 5 *Suppose Assumptions 9- 12 hold. The type-specific CCPs, $p_t(a_t|x_t, \tau)$ are uniquely determined by the joint distribution $\Pr(a_{t+l}, ..., a_t, x_{t+1}, x_t, a_{t-1}, ..., a_{t-l})$.*

Once type-specific CCPs $p_t(a_t|x_t, \tau)$ are identified, we can proceed to identify both utility and subjective expectations for each type of agent using the results in Sections 3 and 4. The heterogeneity of agents can be in one of the three scenarios: they have different subjective expectations, or preferences, or both. Our identification procedure allows us to recover agent utility functions and subjective expectations for each type, thus we are able to distinguish the three scenarios.

6 Estimation and Monte Carlo Evidence

In this section, we first discuss the estimation of DDC models with subjective expectations. Then we present some Monte Carlo evidence of our proposed estimators.

6.1 Estimation

Our identification result provides a closed-form solution to the agent subjective expectations for both finite and infinite time horizons. One may follow the identification procedure to estimate the subjective expectations by a closed-form estimator. Agent preferences then can be estimated in a second step using the CCP approach based on Hotz and Miller (1993). Since the closed-form estimator involves the inversion of matrices, its performance would be unstable if these matrices are near singular. As an alternative, we propose a maximum likelihood estimator to estimate subjective expectations and agent preferences in one step.

Suppose we observe in the data n agent actions for T periods, together with the states, denoted as $\{a_{it}, x_{it}\}_{it}$, where $i = 1, ..., n$ and $t = 1, ..., T$. We denote the parameters in payoff functions, objective transitions, and subjective expectations as θ_u, θ_o, and θ_s, respectively. We first present the likelihood function of the data $\{a_{it}, x_{it}\}_{it}$.

$$\begin{aligned} & \mathcal{L}(x_2, ..., x_T, a_1, ...a_T | x_1; \theta_u, \theta_s, \theta_o) \\ = \ & \Pi_{i=1}^n \Pi_{t=2}^T p_t(a_{it}|x_{it}; \theta_u, \theta_s) f(x_{it}|x_{i,t-1}, a_{it-1}; \theta_o) p_1(a_{i1}|x_{i1}; \theta_u, \theta_s), \end{aligned}$$

where $p_t(a_{it}|x_{it}; \theta_u, \theta_s)$ is the set of agent CCPs in period t. The log-likelihood function is additively separable in the following two components:

$$L \equiv \log \mathcal{L} = \sum_{i=1}^n \sum_{t=1}^T \log p_t(a_{it}|x_{it}; \theta_u, \theta_s) + \sum_{i=1}^n \sum_{t=2}^T \log f(x_{it}|x_{it-1}, a_{it-1}; \theta_o). \quad (25)$$

Thus, we can estimate preferences θ_u and subjective expectations θ_s separately from estimating objective transition θ_u. That is, the parameters (θ_u, θ_s) and θ_o can be estimated by maximizing the first and second parts of the above log-likelihood function, respectively. We use $\hat{\theta}$ to denote the estimator of the corresponding parameters.

Recall that some elements of subjective expectations have to be normalized for identification. Without loss of generality, we assume that agents have rational expectations about some state transition so that we can use the objective transition of these states in the estimation. Under such normalization, we divide the parameters associated with subjective expectations into two parts: $\theta_s \equiv (\theta_s^n, \theta_s^e)$, where θ_s^n and θ_s^e are the parameters to be normalized and estimated, respectively. θ_s^n can be obtained from the estimated objective state transition $\hat{\theta}_o$, denoted as $\hat{\theta}_s^n$. The parameters of preferences and subjective expectations, $\theta \equiv (\theta_u, \theta_s^e)$, can be estimated through the following maximization,

$$\max_\theta \sum_{i=1}^n \sum_{t=1}^T \log p_t(a_{it}|x_{it}; \theta, \hat{\theta}_s^n). \quad (26)$$

Since the $p_t(a_{it}|x_{it}; \theta, \theta_s^n)$ representation of CCPs for finite horizon differs from that for infinite horizon, we present the estimators for both scenarios separately.

Finite horizon. For a finite horizon model, we use backward induction to back out the CCPs for each period t. To do so, we start in the terminal period T, in which the optimal behaviors depend on the specification of the continuation value. If we assume that the continuation value is zero in the terminal period, the choice-specific value function is the same as the per-period flow utility. Then, we move to period $T - 1$ and continue the procedure until we reach the first period in our dataset.

Infinite horizon. In the case of infinite horizon, the dynamic framework is stationary so that the value function and CCPs are not indicated by time. Furthermore, for a given set of CCPs, the ex ante value function can be solved from a system of fixed point equations, represented as the value operator $\phi(\cdot)$. Plugging this ex ante value function into equation (2) for all actions and states, we can further represent the CCPs as a fixed point in the following mapping,

$$p = \Psi(p; \theta_u, \theta_s) \equiv \Lambda(\phi(p; \theta_u, \theta_s)), \tag{27}$$

where p collects CCPs for all actions and state, and $\Psi(\cdot)$ represents the fixed point mapping of the $J(K+1) \times 1$ vector of CCPs p. This fixed point mapping $\Psi(\cdot)$ first maps p into the $J \times 1$ ex ante value function through the value operator $\phi(\cdot)$; the vector of value function is then mapped to CCPs following equation (2) collected to be the policy operator $\Lambda(\cdot)$.

To estimate θ, we adopt a Nested Pseudo Likelihood Algorithm (NPL) proposed in Aguirregabiria and Mira (2002). To implement the algorithm, we start with an initial guess p^0 for the CCPs. At the τ-th ($\tau \geq 1$) step of iteration, our estimation takes the following two steps.

- Step 1: given p^τ, we obtain a pseudo-likelihood estimate of θ, $\hat{\theta}^\tau$, which satisfies

$$\hat{\theta}^\tau = \arg\max_\theta \sum\nolimits_{i=1}^{n} \sum\nolimits_{t=1}^{T} \log p^\tau(a_{it}|x_{it}; \theta, \hat{\theta}_s^n),$$

  where $p^\tau(a_{it}|x_{it}; \theta)$ is an element of p^τ satisfying the mapping $p^\tau = \Psi(p^\tau; \theta, \hat{\theta}_s^n)$.

- Step 2: we update the CCPs using the estimated parameters $\hat{\theta}^\tau$ from the mapping

$$p^{\tau+1} = \Psi(p^\tau; \theta^\tau, \hat{\theta}_s^n).$$

We iterate the two steps above until p and θ converge. We refer to Kasahara and Shimotsu (2008) for convergence of the estimator generated from the NPL algorithm to the Maximum likelihood Estimation (MLE). In particular, the NPL estimator converges to the MLE estimator at a super-linear, but less-than-quadratic, rate.

The difference between our estimator and those in the existing literature lies in the role of subjective expectations in the estimation. Specifically, we estimate part of the subjective expectations θ_s^e together with the payoff primitives θ_u. By contrast, the existing literature assumes that the subjective expectations are the same as the objective state transitions, i.e., $\theta_s = \theta_o$, and then estimates θ_s directly from data in the first step and the payoff primitives θ_u in the second step.

6.2 Monte Carlo Experiments

In this section, we present Monte Carlo results to illustrate the finite sample performance of the proposed estimators. The Monte Carlo experiments are conducted for finite and infinite horizon cases separately.

We consider a binary choice DDC model in both finite and infinite horizon scenarios. First, we set up the payoff primitives, the objective law of motion, and agent expectations about the transition of the state variable, where the agent may or may not have perfect expectations. Given these primitives, we solve for agent CCPs, either by backward induction or by contraction mapping, depending on the horizon of the model. We then use the CCPs and objective transition matrices to simulate agent actions and states, respectively. Next, we estimate the parameters of interest, following the estimators proposed in the previous section. The objective transition matrices are estimated using MLE, and the payoff primitives and subjective expectations are estimated together using MLE and NPL estimators in finite and infinite horizon cases, respectively. When data are generated from subjective expectations, we also estimate the payoff primitives by imposing the assumption of rational expectations, as in the existing literature, for comparison.

In the finite horizon case, the per-period utility function is specified as follows.

$$u(a,x) = \begin{cases} \epsilon_0, & \text{if } a = 0, \\ u(x) + \epsilon_1, & \text{if } a = 1, \end{cases}$$

where ϵ_0 and ϵ_1 are drawn from a mean-zero type-I extreme value distribution. Note that the continuation value at the terminal period is assumed to be zero. We set $J = 3$, i.e., the state variable x takes three values, $x \in \{x_1, x_2, x_3\}$ so we have three utility parameters: $u(x_1) = -2, u(x_2) = 0.4, u(x_3) = 2.1$. The objective law of motion for the state variable x conditional on choice $a = 0$ and $a = 1$, is represented by the following 3×3 matrices

$$\boldsymbol{TR}_0 = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.19 & 0.71 \end{bmatrix}; \qquad \boldsymbol{TR}_1 = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.5 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix},$$

respectively. Let $\boldsymbol{S}_0$ and $\boldsymbol{S}_1$ denote subjective expectations on the law of motion for action 0 and 1, respectively. We consider three different scenarios for the expectations: (1) Agents have perfect expectations, i.e., $\boldsymbol{S}_1 = \boldsymbol{TR}_1$ and $\boldsymbol{S}_0 = \boldsymbol{TR}_0$. (2) Agents' subjective expectations about the state transition associated with action $a = 1$ are the same as their objective counterparts, i.e., $\boldsymbol{S}_1 = \boldsymbol{TR}_1$; agents' subjective expectations about the state transition associated with action $a = 0$ deviates from their objective counterparts and is expressed in the following matrix:

$$\boldsymbol{S}_0 = \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.1 & 0.8 & 0.1 \\ 0.05 & 0.095 & 0.855 \end{bmatrix}.$$

(3) Agents' subjective expectations on the transition of one state $x = J$ associated with action $a = 1$ are the same as their objective counterparts, i.e., $\boldsymbol{S}_1(3) = \boldsymbol{TR}_1(3)$; the expectations on the remaining law of motion deviate from their counterparts and are

expressed as follows:

$$\boldsymbol{S}_0 = \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.1 & 0.8 & 0.1 \\ 0.05 & 0.095 & 0.855 \end{bmatrix}, \qquad \boldsymbol{S}_1 = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.25 & 0.6 & 0.15 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}.$$

Settings (2) and (3) correspond to Assumptions 6A and 6B, respectively. In these two settings, identification requires $J+1=4$ and $2*J=6$ periods of data, respectively. For comparability of estimator performance across the two settings, we simulate the data for $2*J=6$ periods, regardless of the setting.

For the infinite horizon setting, agent choice is binary and there are two state variables, x and w. Based on the identification results, the payoff function is assumed as follows

$$u(a,x,w) = \begin{cases} \epsilon_0, & \text{if } a=0; \\ u^1(x)+u^2(x)w+\epsilon_1, & \text{if } a=1, \end{cases} \tag{28}$$

where ϵ_0 and ϵ_1 are drawn from a mean-zero type-I extreme value distribution. Both state variables are assumed to be discrete: $x \in \{x_1, x_2\}$ and $w \in \{w_1, w_2, w_3, w_4\}$. The utility parameters are $u^1(x_1)=0.1, u^1(x_2)=0.2, u^2(x_1)=0.2$, and $u^2(x_2)=-0.2$. The objective state transition processes for x, $\boldsymbol{TR}_0^x$, and $\boldsymbol{TR}_1^x$, and for w, $\boldsymbol{TR}_0^w$, and $\boldsymbol{TR}_1^w$ are setup as follows.

$$\boldsymbol{TR}_0^w = \begin{bmatrix} 0.6 & 0.2 & 0.2 & 0 \\ 0.1 & 0.75 & 0.15 & 0 \\ 0.04 & 0.1 & 0.8 & 0.06 \\ 0.01 & 0.08 & 0.1 & 0.81 \end{bmatrix}; \quad \boldsymbol{TR}_1^w = \begin{bmatrix} 0.7 & 0.1 & 0.15 & 0.05 \\ 0.2 & 0.65 & 0.05 & 0.1 \\ 0.04 & 0.01 & 0.9 & 0.05 \\ 0.02 & 0.18 & 0.1 & 0.7 \end{bmatrix};$$
$$\boldsymbol{TR}_0^x = \begin{bmatrix} 0.6 & 0.4 \\ 0.45 & 0.55 \end{bmatrix}; \qquad \boldsymbol{TR}_1^x = \begin{bmatrix} 0.1 & 0.9 \\ 0.5 & 0.5 \end{bmatrix},$$

where the subscripts 0 and 1, respectively, represent $a=0$ and $a=1$.

Agents have rational expectations about the transition of w but may have subjective expectations about the transition of x. We consider two settings for Monte Carlo experiments: (1) Agents have rational expectations; that is, agent expectations about the state evolution are the same as their objective counterparts, i.e., $\boldsymbol{S}_a^w = \boldsymbol{TR}_a^w$, $\boldsymbol{S}_a^x = \boldsymbol{TR}_a^x$, $a \in \{0,1\}$. (2) Agents' subjective expectations satisfy Assumption 6B, and $\boldsymbol{S}_a^w = \boldsymbol{TR}_a^w, a \in \{0,1\}$ and $\boldsymbol{S}_1^x = \boldsymbol{TR}_1^x$, while $\boldsymbol{S}_0^x \neq \boldsymbol{TR}_0^x$, where

$$\boldsymbol{S}_0^x = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}.$$

We conduct Monte Carlo experiments for every scenario, using sample sizes $n = 300, 600, 1000, 2500$, and standard errors are estimated from 1000 replications. Before estimation, we check full rank Assumptions 4 and 5A for our simulated samples and find that the assumptions hold.

The results of the Monte Carlo experiments for the finite horizon case are presented in Tables 1-4, and for the infinite horizon case in Tables 5-6. We draw three main findings. First, the proposed estimator performs well across different settings for moderate sample sizes. More importantly, as shown in Tables 1, 3, and 5, our estimates track the true parameters closely, even when the data are generated from a model with rational expectations. Not surprisingly, in such cases, the standard errors of our estimates are generally larger than those from imposing the restriction of rational expectations. Second, failing to account for subjective expectations may lead to significant estimation bias. When data are generated from subjective expectations, the parameters of the utility function estimated by imposing rational expectations are off the true values. This can be seen in Tables 2 and 4, where estimates of $u(x_1), u(x_2)$, and $u(x_3)$ are significantly different from the true parameters. The differences persist as sample size increases from 300 to $2,500$.

7 Empirical Illustration: Women's Labor Force Participation

In this section, we apply the proposed method to the Panel Study of Income Dynamics (PSID) data and study women's labor force participation. Female labor supply has been extensively studied in the literature; e.g., see Eckstein and Wolpin (1989) and Blundell, Costa Dias, Meghir, and Shaw (2016), among others. Instead of providing a thorough analysis of women's labor force participation, we should emphasize that this section is an illustrative application of the identification and estimation method for the subjective in the previous section.

We assume that a woman and her husband jointly make a decision on her labor force participation. Their expectations about the evolution of their household income affect the wife's labor participation decisions. Our main objective is twofold: (1) to investigate whether their expectations deviate from rational expectations, and if yes, (2) to conduct a counterfactual analysis and analyze how women's decisions would change if they were to have rational expectations.

We follow Eckstein and Wolpin (1989) to make some key working assumptions in our analysis. First, we simplify the choice of hours of work to a binary working/non-working decision. Second, we ignore the husband's labor force participation decisions. Third, we focus on women above the age 39 to avoid modeling fertility decisions. Finally, marriage is taken as exogenously given.

7.1 Data

The PSID is a longitudinal survey consisting of a nationally representative sample of over 18,000 individuals living in 5,000 families in the United States. The original sample was re-interviewed annually starting from 1968 to 1997 and biennially thereafter. The PSID collects data on annual income and female labor force participation for the preceding calendar year. We only use data collected up to 1997, since our identification strategy relies on variation in CCPs over consecutive years.

We construct an annual employment profile for each woman between the age of 39 and 60, where 60 is assumed to be the terminal period of a woman's labor participation decision because women are typically out of the labor force by age 60, e.g., Eckstein and Wolpin (1989). The number of years observed in the data varies across women. Table 7 summarizes the frequency of observations by years. As shown in the table, the 1,673 women in our sample are not evenly distributed across years. Almost 34% of them appear in all 17 years of data and more than half of them appear in over 13 years of data. Table 8 presents the summary statistics of our sample, where we aggregate the information of those women who are observed in at least six years (for the purpose of identification) and the income is expressed in 1999 dollars. The average household income was $57,700 with relatively large variation across households and years. The majority of the women in our sample were between 42 ad 55 years of age, and the average educational attainment was high school. On average (at the individual-by-year level), 58% of these women were employed at the time of the survey.

7.2 Model specification

Each household in the sample is assumed to maximize its present value of utility over a known finite horizon by choosing whether or not the wife works in each discrete time period, assuming a discount factor of $\beta = 0.95$. This framework fits into a finite horizon model. A household's flow utility function is assumed to be stationary and is specified as

$$u(a, x, \epsilon) = \begin{cases} \epsilon_0, & \text{if } a = 0, \\ u(x) + \epsilon_1, & \text{if } a = 1, \end{cases}$$

where ϵ_0 and ϵ_1 are drawn from a mean-zero type-I extreme value distribution and are assumed to be independent over time, a is a binary variable that equals 1 if the wife works during period t, and 0 otherwise; x is the household income.

Analysis of finite horizon DDCs requires specification of continuation values in the terminal period. In the existing literature, a simple approach is to assume the continuation value in the terminal period to be zero, then the choice-specific value function in the terminal period is the same as the stationary flow utility. This assumption simplifies estimation as there is no need to estimate the additional predetermined continuation value for the terminal period. However, such an assumption is not appropriate in our application because agents are alive and their continuation value is likely to be nonzero after they stop working. To better describe agents' decisions, we allow the continuation value for the terminal period to be nonzero and specify the choice-specific value function in the terminal period as:

$$\tilde{u}(a, x, \epsilon) = \begin{cases} \epsilon_0, & \text{if } a = 0, \\ \tilde{u}(x) + \epsilon_1, & \text{if } a = 1. \end{cases}$$

$u(x) = \tilde{u}(x)$ is equivalent to a zero continuation value for the terminal period (See Arcidiacono and Ellickson (2011)).

Note that, while income is usually continuous, our identification is for discrete state space. Therefore, we make several simplifications in this empirical application. We discretize the observed household income into three values ($J = 3$).

$$x = \begin{cases} 1, & \text{if household income} \leq \$17,000, \\ 2, & \text{if } \$17,000 < \text{household income} \leq \$150,000, \\ 3, & \text{if household income} > \$150,000, \end{cases}$$

where $x = 1, 2, 3$ are referred to as low, medium, and high income, respectively. The cutoff \$17,000 is roughly the U.S. Department of Health & Human Services (HHS) poverty line for a family size of 4, while \$150,000 is the income level that leads to a "good" life in America according to a survey of WSL Strategic retail.[8]

We impose two assumptions to identify the model. First, we assume that subjective expectations are homogenous, even though women in our sample differ in age and education.[9] We believe that homogeneity of subjective expectations is a reasonable first-order approximation because we only focus on women between ages 39 to 60, and arguably they are sufficiently experienced such that age and eduction would not significantly affect their subjective expectations. Second, we assume that subjective expectations about the future income distribution for those high-income households with a working wife, i.e., $s(x'|x = 3, a = 1)$, is known to be the same as the objective transition observed in the data. This normalization is required for identification as stated in Assumption 6A. We impose this restriction because the future income for a high-income household with a working wife is less uncertain. For example, high-income households may not qualify for some social welfare programs as low-income households do, so it is relatively easier for high-income households to predict their future income if the wife works.

7.3 Estimation results

To estimate the model, we use the setting described in Theorem 1, where identification requires at least $2J = 6$ periods of observations. Thus, for our sample, we retain who appear in at least six periods for our estimation.

Table 9 presents the estimation results of subjective and rational expectations, and the corresponding preferences.[10] The top panel provides estimates of transition matrices and parameters of utility under subjective expectations. For comparison, the bottom panel displays estimates obtained by imposing the assumption of rational expectations,

[8] Rosenberg, Yuval (2012, March 7), *The Fiscal Times.* Retrieved from https://www.businessinsider.com/the-basic-annual-income-every-american-would-need-is-150000-2012-3.

[9] We can easily incorporate heterogenous subjective expectations in our analysis. However, such an approach would require a larger sample size.

[10] Before estimating the model parameters, we test the rank of the observed matrix $\Delta\boldsymbol{\xi}_{i,K}$ and find that it is full rank, i.e., Assumption 5A holds.

for which the parameters are estimated using maximum likelihood directly on the data. Table 9 reveals discrepancies between subjective expectations and their rational counterparts about income transitions. We formally test whether the two sets of expectations are the same, i.e., whether households hold rational expectations about income transitions. Conditional on both non-working and working status, we reject the null hypothesis that subjective and rational expectations are equal at the 1% significance level. The test result indicates that agents do not have perfectly foresight about income transition, this implies that our understanding of agents' working decisions under the assumption of rational expectations could be misleading. We also test whether households' subjective expectations vary conditional on the wife's working status and find that subjective expectations of households with a working wife differ significantly from those with a non-working wife at the 1% significance level. We further conduct such a test for the objective transitions of income and obtain similar results. The results demonstrate that agents are sophisticate enough to predict the different impacts of their actions on income transitions, even though they do not have rational expectations.[11]

There are two important observations from the results in Table 9. First, households with a non-working wife are overly pessimistic about their income transitions, while those with a working wife are less so. For example, among medium-income households with a non-working wife, agents expect their household income to stay in the medium category with probability 0.85 and drop to the low income category with probability 0.15. By contrast, the objective transition probabilities for income staying as medium, dropping to low, and increasing to high are 0.92, 0.07, and 0.01, respectively. A similar pattern can be seen for households of low and high income. On the other hand, for households with a working wife, subjective expectations are very close to the objective transition probabilities, even though rational expectations are rejected as we described above. This finding demonstrates that deviation of subjective expectations from rational expectations depends on agent actions. Since investigation of expectation formation is out of scope of this paper, and we leave this to future work.

Second, agents have "asymmetric" expectations about their income transitions. In households with a non-working wife, agents of medium-income households believe that income will remain as medium with probability 0.85, which is about 0.07 smaller than the corresponding objective probability. However, agents of high-income households are more

[11] In the empirical illustration, we also test stationarity of subjective expectations by assuming that women have two sets of subjective beliefs, one at age 39-48 while the other one at age 49-59. This implicitly assumes that learning occurs once. We also assume that women are not aware of the learning when they make decisions before age 49. We estimate the flow utility, continuation value in the ending period, and the two different sets of subjective expectations altogether. Note that normalization is required for each set of subjective expectations, and we use the objective transition in the same phase to pin down the subjective expectations for high-income households with a working wife, i.e., $s(x'|x = 3, a = 1)$. We then test whether the subjective expectations are stationary, i.e., subjective expectations are the same in the two phases. Our testing results show that at the 1% significance level, we fail to reject the null that subjective expectations are stationary. We stress, however, that these findings are based on the *ad hoc* assumption that women are unaware of the change of their subjective expectations after age 48. While these findings are useful for us to justify Assumption 2(b), one should be cautiously in using the results as evidence against learning.

pessimistic: they believe with almost certainty that income will drop to medium, while the objective probability of this drop is just 0.24. This finding is consistent with survey data suggesting asymmetric expectations of agents. For example, Heimer, Myrseth, and Schoenle (2018) find discrepancies between surveyed mortality expectations and actuarial statistics from the Social Security administration, and these discrepancies differ across age groups.

Estimated preferences under subjective expectations and rational expectations share a similar pattern: women prefer to work if their household income is medium or high while they prefer not to work if their current income is low. This may be due to the possibility that women in low-income households likely have lower educational attainment and consequently face less attractive job options, which may explain her reluctance to work. Our estimates of agent preferences indicate that the utility of the terminal period is different from the flow utility under both rational and subjective expectations. This implies that the continuation value after the terminal period is not zero, at least for this data set.

Next, we investigate how the discrepancies between subjective and rational expectations affect women's labor force participation. For this purpose, we conduct a counterfactual analysis by using the estimates in Table 9 to simulate CCPs and compute percentage differences under subjective and rational expectations.[12] The results presented in Table 10 suggest (1) having rational expectations would decrease labor participation; and (2) the impact of subjective expectations on labor participation choice is heterogenous. Regardless of household income level, women with rational expectations would be less likely to work, but this disparity decreases as the women approach age 60. The difference in CCPs between the two sets of expectations for women in low-income households is about three times of that of medium and high-income categories. For example, at age 57, the percentage difference of CCPs for low, medium and high-income categories are 12.8%, 4.4%, and 3.7%, respectively. This counterfactual analysis has important policy implications. If a government aims to promote labor force participation among women, providing them with accurate information about income transitions would not work achieve the desired result as choice are already oversupplying labor due to their subjective expectations.

8 Concluding Remarks

This paper studies dynamic discrete choice models with agents holding subjective expectations about the state transition, where these expectations may be different from the objective transition probabilities observed in the data. We show that agents' subjective expectations and preferences are nonparametrically identified in both finite and infinite horizon cases. The identification power in finite horizon cases comes from variation in CCPs across time and potentially an additional exclusive variable, while the identifica-

[12]Note that these counterfactuals, in general, are not invariant to the normalization of the utility function. See, e.g., Norets and Tang (2013), Kalouptsidi, Scott, and Souza-Rodrigues (2015), and Arcidiacono and Miller (2013).

tion power in infinite horizon cases comes from the information of an additional state variable. We propose to estimate the model using a maximum likelihood estimator, and we present Monte Carlo evidence illustrating that our estimator performs well with mid-sized samples. Applying our methodology to PSID data, we illustrate that households do not hold rational expectations about their income transitions, and the discrepancies between their subjective expectations and the objective transition probabilities may lead to a significant difference in women's labor force participation. Our estimates also shed light on how subjective expectations affects agents' dynamic decisions and what policies would be appropriate for improving labor market participation among women.

A direction for future research is to relax some important assumptions in this paper, e.g., invariant subjective expectations, and to incorporate learning into the model. While our method is introduced in the context of discrete choice, it may be possible to extend it to dynamic models with continuous choice, e.g., life-cycle consumption problems. We are considering these possibilities for future work.

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Appendix

A Proofs

This section provides proofs for all the identification results.

A.1 Proof of Theorem 1

The proof of Theorem 1 is sketched in the main text. Therefore, we only provide proofs for the main steps used to derive Theorem 1.

Derivation of equation (4)

$$\begin{aligned}\sum_{x'=1}^{J} V_{t+1}(x')s(x'|x,a) &= \sum_{x'=1}^{J-1} V_{t+1}(x')s(x'|x,a) + \beta V_{t+1}(J)\big[1 - \sum_{x'=1}^{J-1} s(x'|x,a)\big] \\ &= \sum_{x'=1}^{J-1} \big[V_{t+1}(x') - V_{t+1}(J)\big]s(x'|x,a) + \beta V_{t+1}(J). \\ &\equiv S_a(x)\boldsymbol{V}_{t+1} + \beta V_{t+1}(J), \qquad \text{(A.1)}\end{aligned}$$

Consequently, the log ratio of CCPs for any t can be represented as:

$$\begin{aligned}\xi_{t,i,K}(x) &\equiv \log\left(\frac{p_{t,i}(x)}{p_{t,K}(x)}\right) = \big[u_i(x) - u_K(x)\big] + \beta \sum_{x'} V_{t+1}(x')\big[s(x'|x,i) - s(x'|x,K)\big] \\ &= u_i(x) - u_K(x) + \beta\big[S_i(x) - S_K(x)\big]\boldsymbol{V}_{t+1}. \qquad \text{(A.2)}\end{aligned}$$

Derivation of equation (7) We derive the recursive relationship of value functions over time. We first prove that we can express the *ex-ante* value function using the choice-specific value function $v_{t,K}(x)$ with an adjustment of $-\log p_{t,K}(x)$.

In the model we consider, the *ex-ante* value function at t can be expressed as

$$\begin{aligned}V_t(x) &= \int \sum_{a\in\mathcal{A}} \mathbb{1}\{a = a_t\}\big[v_t(x_t,a) + \epsilon_t(a)\big] f(\epsilon_t)d\epsilon_t \\ &= \log\left\{\sum_{a\in\mathcal{A}} \exp\big[v_t(x,a)\big]\right\},\end{aligned}$$

where the second equality is obtained under the assumption that ϵ_t is distributed according to a mean zero type-I extreme value distribution. The conditional choice probability for $K \in \mathcal{A}$ is

$$p_t(a = K|x) = \frac{\exp\big[v_t(x,K)\big]}{\sum_{a\in\mathcal{A}} \exp\big[v_t(x,a)\big]}.$$

We further simplify $V_t(x)$ as follows.

$$\begin{aligned} V_t(x) &= -\log p_t(a=K|x) + v_t(x, a=K) \\ &\equiv -\log p_{t,K}(x) + v_{t,K}(x). \end{aligned}$$

Next, we build up a link between the *ex-ante* value function and agents' subjective expectations based on the equation above.

$$\begin{aligned} V_t(x) &= -\log p_{t,K}(x) + v_{t,K}(x) \\ &= -\log p_{t,K}(x) + u_K(x) + \beta \sum\nolimits_{x'} V_{t+1}(x')s(x'|x,a) \\ &= -\log p_{t,K}(x) + u_K(x) + \beta S_K(x)\boldsymbol{V}_{t+1} + \beta V_{t+1}(J). \end{aligned}$$

Consequently, the *ex-ante* value function for state $x \in \mathcal{X}, x \neq J$ relative to J in period t can be represented as

$$\begin{aligned} V_t(x) - V_t(J) &= \big[-\log p_{t,K}(x) + u_K(x) + \beta S_K(x)\boldsymbol{V}_{t+1} + \beta V_{t+1}(J) \big] \\ &- \big[-\log p_{t,K}(J) + u_K(J) + \beta S_K(J)\boldsymbol{V}_{t+1} + \beta V_{t+1}(J) \big] \\ &= -\big[\log p_{t,K}(x) - \log p_{t,K}(J) \big] + \big[u_K(x) - u_K(J) \big] + \beta \big[S_K(x) - S_K(J) \big] \boldsymbol{V}_{t+1}. \end{aligned}$$

We stack the equation above for all state $x, x \neq J$ and obtain the following matrix form.

$$\begin{aligned} \boldsymbol{V}_t &\equiv \begin{pmatrix} V_t(1) - V_t(J) \\ V_t(2) - V_t(J) \\ \ldots \\ V_t(J-1) - V_t(J) \end{pmatrix} \\ &= -\begin{pmatrix} \log \boldsymbol{p}_{t,K}(1) - \log \boldsymbol{p}_{t,K}(J) \\ \log \boldsymbol{p}_{t,K}(2) - \log \boldsymbol{p}_{t,K}(J) \\ \ldots \\ \log \boldsymbol{p}_{t,K}(J-1) - \log \boldsymbol{p}_{t,K}(J) \end{pmatrix} + \begin{pmatrix} u_K(1) - u_K(J) \\ u_K(2) - u_K(J) \\ \ldots \\ u_K(J-1) - u_K(J) \end{pmatrix} + \beta \begin{pmatrix} S_K(1) - S_K(J) \\ S_K(2) - S_K(J) \\ \ldots \\ S_K(J-1) - S_K(J) \end{pmatrix} \boldsymbol{V}_{t+1} \\ &\equiv -\log \boldsymbol{p}_{t,K} + \boldsymbol{u}_K + \widetilde{\boldsymbol{S}}_K \boldsymbol{V}_{t+1}, \end{aligned}$$

where $\log \boldsymbol{p}_{t,K}$ and $\boldsymbol{u}_K$ are defined similarly to $\boldsymbol{V}_t$, and $\widetilde{\boldsymbol{S}}_K \equiv \big[S_K(1) - S_K(J), S_K(2) - S_K(J), \cdots, S_K(J-1) - S_K(J)\big]'$. Consequently, we have the following recursive connection for *ex-ante* value functions over time.

$$\boldsymbol{V}_t = -\log \boldsymbol{p}_{t,K} + \boldsymbol{u}_K + \widetilde{\boldsymbol{S}}_K \boldsymbol{V}_{t+1}.$$

Derivation of equation (8) Taking the first difference in equation (7) for all the terms, we have

$$\Delta \boldsymbol{V}_t = -\Delta \log \boldsymbol{p}_{t,K} + \beta \widetilde{\boldsymbol{S}}_K \boldsymbol{\Delta V}_{t+1}, \tag{A.3}$$

where $\Delta \boldsymbol{V}_t \equiv \boldsymbol{V}_t - \boldsymbol{V}_{t-1}$, and $\Delta \log \boldsymbol{p}_{t,K}$ is defined analogously.

Derivation of equation (11) Equation (9) holds for all $t = 2, \cdots, T$, then we have

$$\begin{aligned} \boldsymbol{\Delta V}_{t+1} &= \beta^{-1}\big[\boldsymbol{S}_i - \boldsymbol{S}_K\big]^{-1}\boldsymbol{\Delta\xi}_{t,i,K}, \\ \boldsymbol{\Delta V}_t &= \beta^{-1}\big[\boldsymbol{S}_i - \boldsymbol{S}_K\big]^{-1}\boldsymbol{\Delta\xi}_{t-1,i,K}. \end{aligned} \tag{A.4}$$

Plug the equation above into (A.3), we have

$$\beta^{-1}\big[\boldsymbol{S}_i - \boldsymbol{S}_K\big]^{-1}\boldsymbol{\Delta\xi}_{t-1,i,K} = -\Delta\log\boldsymbol{p}_{t,K} + \widetilde{\boldsymbol{S}}_K\big[\boldsymbol{S}_i - \boldsymbol{S}_K\big]^{-1}\boldsymbol{\Delta\xi}_{t,i,K}. \tag{A.5}$$

Consequently,

$$\Big[\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}, \quad -\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Big]\begin{bmatrix} \boldsymbol{\Delta\xi}_{t,i,K} \\ \boldsymbol{\Delta\xi}_{t-1,i,K} \end{bmatrix} = \Delta\log\boldsymbol{p}_{t,K}, \tag{A.6}$$

Identification of expectations associated with other actions. In what follows, we show that the subjective expectations associated with other actions i', $i' \neq i$ and$i' \neq K$, can be identified. We augment equation (6) to a matrix equation

$$\Delta\widetilde{\boldsymbol{\xi}}_{i,K} = \beta[\boldsymbol{S}_i - \boldsymbol{S}_K]\Delta\widetilde{\boldsymbol{V}}, \tag{A.7}$$

where $\Delta\widetilde{\boldsymbol{\xi}}_{i,K} = \Big[\Delta\xi_{\tau_1,i,K}, \Delta\xi_{\tau_2,i,K}, ..., \Delta\xi_{\tau_{J-1},i,K}\Big]$ and $\Delta\widetilde{\boldsymbol{V}} = \Big[\Delta\boldsymbol{V}_{\tau_1+1}, \Delta\boldsymbol{V}_{\tau_2+1}, ..., \Delta\boldsymbol{V}_{\tau_{J-1}+1}\Big]$. The time periods $\tau_1, \tau_2, \cdots, \tau_{J-1}$ are chosen such that $\Delta\widetilde{\boldsymbol{\xi}}_{i,K}$ is invertible. Because the matrix $\Delta\widetilde{\boldsymbol{\xi}}_{i,K}$ is constructed through reducing the dimension of the invertible matrix $\Delta\boldsymbol{\xi}_{i,K}$, the existence of such $\{\tau_1, \tau_2, \cdots, \tau_{J-1}\}$ is guaranteed by Assumption 5A. The invertibility of $\Delta\widetilde{\boldsymbol{\xi}}_{i,K}$, together with Assumption 4 allows us to identify $\Delta\widetilde{\boldsymbol{V}}$ as $\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Delta\widetilde{\boldsymbol{\xi}}_{i,K}$, which is also invertible. Moreover, equation (A.7) holds for all other choices $i' \neq i, K$, i.e.,

$$\Delta\widetilde{\boldsymbol{\xi}}_{i',K} = \beta[\boldsymbol{S}_{i'} - \boldsymbol{S}_K]\Delta\widetilde{\boldsymbol{V}}. \tag{A.8}$$

This equation and the identified matrix $\Delta\widetilde{\boldsymbol{V}}$ enable us to identify the subjective expectations associated with choice i' as follows.

$$\begin{aligned} \boldsymbol{S}_{i'} &= \beta^{-1}\Delta\widetilde{\boldsymbol{\xi}}_{i',K}\Delta\widetilde{\boldsymbol{V}}^{-1} + \boldsymbol{S}_K \\ &= \Delta\widetilde{\boldsymbol{\xi}}_{i',K}\Delta\widetilde{\boldsymbol{\xi}}_{i,K}^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K] + \boldsymbol{S}_K. \end{aligned} \tag{A.9}$$

A.2 Proof of Theorem 2

First we collect all variations of CCP log ratios in the following matrix representation:

$$\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Delta\boldsymbol{\xi}_{i,K}^1 - \beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Delta\boldsymbol{\xi}_{i,K}^2 = \Delta\log\boldsymbol{p}_K.$$

Note that when the subjective expectation matrix associated with action K is known, we can rewrite the equation above in the following vectorization expression,

$$\begin{aligned} vec(\Delta \log \boldsymbol{p}_K) &= vec(\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Delta\boldsymbol{\xi}^1_{i,K}) - vec(\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Delta\boldsymbol{\xi}^2_{i,K}) \\ &= [(\Delta\boldsymbol{\xi}^1_{i,K})' \otimes (\widetilde{\boldsymbol{S}}_K)]vec([\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}) - \beta^{-1}[(\Delta\boldsymbol{\xi}^2_{i,K})' \otimes I]vec([\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}) \\ &= [(\Delta\boldsymbol{\xi}^1_{i,K})' \otimes (\widetilde{\boldsymbol{S}}_K) - \beta^{-1}(\Delta\boldsymbol{\xi}^2_{i,K})' \otimes I]vec([\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}), \end{aligned} \tag{A.10}$$

where $[(\Delta\boldsymbol{\xi}^1_{i,K})' \otimes (\widetilde{\boldsymbol{S}}_K) - \beta^{-1}(\Delta\boldsymbol{\xi}^2_{i,K})' \otimes I]$ is a $(T-2)\cdot(J-1)$ by $(J-1)\cdot(J-1)$ matrix. Identification requires $[(\Delta\boldsymbol{\xi}^1_{i,K})' \otimes (\beta\widetilde{\boldsymbol{S}}_K) - (\Delta\boldsymbol{\xi}^2_{i,K})' \otimes I]$ is of full column rank, which implicitly imposes the restriction $T-2 \geq J-1$. Again, the full rank condition is empirically testable.

A.3 Identification with the terminal period

In this section, we show that the model can be identified using fewer periods of data than that are required in Theorems 1-2 if data on the terminal period are available, i.e., T is the terminal period.

Note that agents do not need to form expectations for the future at the terminal period under the assumption that the continuation value in the terminal period is zero. Thus, CCPs in the terminal period allow us to identify the relative preference $[u_i(x) - u_K(x)]$, whether agents have subjective or rational expectations. Once the flow utility is identified using CCPs in the terminal period, the impact of preference on the log ratio of CCPs is known. Consequently, we do not need to eliminate the utility to identify subjective expectations as in Theorems 1-2. Specifically, we rewrite equation (4) as:

$$\begin{aligned} \eta_{t,i,K}(x) &\equiv \xi_{t,i,K}(x) - [u_i(x) - u_K(x)] \\ &= \beta[S_i(x) - S_K(x)]\boldsymbol{V}_{t+1}, \end{aligned} \tag{A.11}$$

where $\eta_{t,i,K}(x)$ is identified because the utility difference $u_i(x) - u_K(x)$ is identified. We collect all the over-time variations of $\eta_{t,i,K}(x)$ in the following matrix,

$$\boldsymbol{\eta}_{i,K} \equiv \begin{bmatrix} \boldsymbol{\eta}_{2,i,K} & \boldsymbol{\eta}_{4,i,K} & \cdots & \boldsymbol{\eta}_{T,i,K} \\ \boldsymbol{\eta}_{1,i,K} & \boldsymbol{\eta}_{3,i,K} & \cdots & \boldsymbol{\eta}_{T-1,i,K} \end{bmatrix},$$

where $\boldsymbol{\eta}_{t,i,K}$ is defined analogously to the matrix collecting all log ratio of CCPs $\Delta\boldsymbol{\xi}_{t,i,K}$. Similar to Theorems 1-2, we can identify the subjective expectations by imposing a rank condition stated in the following:

Assumption A.3.1 *(a). The number of periods observed is not smaller than* $2J-1$*, i.e,* $T \geq 2J-1$*. (b). The matrix* $\boldsymbol{\eta}_{i,K}$ *is of full row rank.*

Similar to Assumptions 5A and 5B, Assumption A.3.1 is also testable. We summarize the identification result in the following corollary to Theorem 1.

Corollary 1 *Suppose that Assumptions 1–4, A.3.1, and 6A hold. Then the subjective expectations $s(x'|x,a)$ for $x, x' \in \{1, 2, ..., J\}$ and $a \in \{1, 2, ..., K\}$ are identified as a closed-form function of the CCPs, $p_t(a|x)$, for $t = 1, ..., T$, where $T \geq 2J - 1$.*

Corollary 1 shows that $2J - 1$ periods of data (versus $2J$ periods required in Theorem 1) are sufficient for identification if the terminal period of data are available.

Analogously, if the terminal period is observed and Assumption 6B is imposed, i.e., $\boldsymbol{S}_K$ is known, we can improve upon Theorem 2 by identifying the model with $J-1$ periods of data. We provide some brief discussions on the identification as the procedure is similar to that of Corollary 1 . First of all, utility function can be recovered from the choice in the last period. Using this information and the known subjective expectations $\boldsymbol{S}_K$, we can identify value function $\boldsymbol{V}_t$ for $t = 1, 2, \cdots, T$. To identify the expectations $\boldsymbol{S}_i$, we only need to use equation (A.11) with $\boldsymbol{S}_i$ being the only unknown. We define

$$\widetilde{\boldsymbol{\eta}}_{i,K} \equiv \Big[\boldsymbol{\eta}_{1,i,K}, \boldsymbol{\eta}_{2,i,K}, ..., \boldsymbol{\eta}_{T,i,K}\Big],$$

which is an observed $(J - 1) \times T$ matrix. A testable full rank condition is necessary for identification.

Assumption A.3.2 *(a). The number of periods observed is not smaller than $J - 1$, i.e., $T \geq J - 1$. (b). The matrix $\widetilde{\boldsymbol{\eta}}_{i,K}$ is of full row rank.*

We present the identification result under Assumption A.3.2 as a corollary to Theorem 2:

Corollary 2 *Suppose that Assumptions 1 - 4, A.3.2, and 6B hold. The subjective expectations $s(x'|x,a)$ for $x, x' \in \{1, 2, ..., J\}$ and $a \in \{1, 2, ..., K\}$, are identified as a closed-form function of the CCPs, $p_t(a|x)$ for $t = 1, ..., T,$, where $T \geq J - 1$*

A.4 Proof of Theorem 3

This section provides all necessary proofs for Theorem 3.

Derivation of equation (13)

$$\begin{aligned}
& \sum\nolimits_{x'=1}^{J} \sum\nolimits_{w'=1}^{M} V_{t+1}(x', w') s(x'|x,a) f(w'|w) \\
= & \sum\nolimits_{x'=1}^{J-1} \sum\nolimits_{w'=1}^{M} V_{t+1}(x', w') s(x'|x,a) f(w'|w) + \beta \sum\nolimits_{w'=1}^{M} V_{t+1}(J, w') f(w'|w) \big[1 - \sum\nolimits_{x'=1}^{J-1} s(x'|x,a)\big] \\
= & \sum\nolimits_{x'=1}^{J-1} \sum\nolimits_{w'=1}^{M} \big[V_{t+1}(x', w') - V_{t+1}(J, w')\big] s(x'|x,a) f(w'|w) + \beta \sum\nolimits_{w'=1}^{M} V_{t+1}(J, w') f(w'|w). \\
\equiv & \sum\nolimits_{w'=1}^{M} S_a(x) V_{t+1}(w') f(w'|w) + \beta \sum\nolimits_{w'=1}^{M} V_{t+1}(J, w') f(w'|w) \\
\equiv & S_a(x) \boldsymbol{V}_{t+1} F(w) + \overline{V}_{t+1,w}, \qquad \text{(A.12)}
\end{aligned}$$

where

$$\begin{aligned}
V_{t+1}(w) &\equiv [V_{t+1}(x=1,w),...,V_{t+1}(x=J-1,w)]' - V_{t+1}(J,w), \\
F(w) &= [f(w'=1|w),...,f(w'=M|w)]', \\
\boldsymbol{V}_{t+1} &\equiv [V_{t+1}(w=1),....,V_{t+1}(w=M)], \\
\overline{V}_{t+1,w} &= \beta \sum\nolimits_{w'=1}^{M} V_{t+1}(J,w')f(w'|w).
\end{aligned}$$

Consequently, the log ratio of CCPs in period t can be represented as:

$$\begin{aligned}
\xi_{t,i,K}(x,w) &\equiv \log\left(\frac{p_{t,i}(x,w)}{p_{t,K}(x,w)}\right) \\
&= \big[u_i(x,w) - u_K(x,w)\big] + \beta \sum\nolimits_{x'} \sum\nolimits_{w'=1}^{M} V_{t+1}(x')\big[s(x'|x,i) - s(x'|x,K)\big] f(w'|w) \\
&= \big[u_i(x,w) - u_K(x,w)\big] + \beta\big[S_i(x) - S_K(x)\big]\boldsymbol{V}_{t+1}F(w). \qquad \text{(A.13)}
\end{aligned}$$

Derivation of equation (16) Under the assumption that $\boldsymbol{S}_i - \boldsymbol{S}_K$ is invertible, we consider (15) for both t and $t+1$,

$$\begin{aligned}
\Delta\boldsymbol{V}_{t+1}\boldsymbol{F}_w &= \beta^{-1}(\boldsymbol{S}_i - \boldsymbol{S}_K)^{-1}\Delta\boldsymbol{\xi}_{t,i,K}, \\
\Delta\boldsymbol{V}_{t}\boldsymbol{F}_w &= \beta^{-1}(\boldsymbol{S}_i - \boldsymbol{S}_K)^{-1}\Delta\boldsymbol{\xi}_{t-1,i,K}. \qquad \text{(A.14)}
\end{aligned}$$

We then follow similar argument to the case without the additional state variable w to derive the recursive relationship of the first differences of *ex-ante* value functions. First of all, we can represent the *ex-ante* value function using the choice-specific value function $v_{t,K}(x,w)$ with an adjustment of the corresponding CCPs, i.e., $-\log p_{t,K}(x,w)$.

$$\begin{aligned}
V_t(x,w) &= -\log p_{t,K}(x,w) + v_{t,K}(x,w) \\
&= -\log p_{t,K}(x,w) + u_K(x,w) + \beta \sum\nolimits_{x'=1}^{J} \sum\nolimits_{w'=1}^{M} V_{t+1}(x',w')s(x'|x,K)f(w'|w) \\
&= -\log p_{t,K}(x,w) + \beta S_K(x)\boldsymbol{V}_{t+1}F(w) + \overline{V}_{t+1,w}.
\end{aligned}$$

The *ex-ante* value relative to the baseline state $x = J$ for any w is

$$\begin{aligned}
&V_t(x,w) - V_t(J,w) \\
=\ & \big[-\log p_{t,K}(x,w) + u_K(x,w) + \beta S_K(x)\boldsymbol{V}_{t+1}F(w) + \overline{V}_{t+1,w}\big] \\
-\ & \big[-\log p_{t,K}(J,w) + u_K(J,w) + \beta S_K(J)\boldsymbol{V}_{t+1}F(w) + \overline{V}_{t+1,w}\big] \\
=\ & -\big[\log p_{t,K}(x,w) - \log p_{t,K}(J,w)\big] + \big[u_K(x,w) - u_K(J,w)\big] + \beta\big[S_K(x) - S_K(J)\big]\boldsymbol{V}_{t+1}F(w)
\end{aligned}$$

We stack $V_t(x,w) - V_t(J,w)$ for $x \in \{1, 2, \cdots, J-1\}$ and all w,

$$\begin{aligned}
\boldsymbol{V}_t &\equiv \begin{pmatrix} V_t(1,1) - V_t(J,1) & V_t(1,2) - V_t(J,2) & ... & V_t(1,M) - V_t(J,M) \\ V_t(2,1) - V_t(J,1) & V_t(2,2) - V_t(J,2) & ... & V_t(2,M) - V_t(J,M) \\ ... & & & \\ V_t(J-1,1) - V_t(J,1) & V_t(J-1,2) - V_t(J,2) & ... & V_t(J-1,M) - V_t(J,M) \end{pmatrix} \\
&\equiv -\log \boldsymbol{p}_{t,K} + \boldsymbol{u}_K + \widetilde{\boldsymbol{S}}_K \boldsymbol{V}_{t+1} \boldsymbol{F}_w,
\end{aligned}$$

where $\log \boldsymbol{p}_{t,K}$ and $\boldsymbol{u}_K$ are analogously defined to $\boldsymbol{V}_t$. Taking the first difference of the equation above, we have the following recursive relationship that is similar to (8)

$$\Delta \boldsymbol{V}_t = -\Delta \log \boldsymbol{p}_{t,K} + \beta \widetilde{\boldsymbol{S}}_K \boldsymbol{\Delta V}_{t+1} \boldsymbol{F}_w, \tag{A.15}$$

where $\Delta \boldsymbol{V}_t \equiv \boldsymbol{V}_t - \boldsymbol{V}_{t-1}$, and $\Delta \log \boldsymbol{p}_{t,K}$ is defined in the same way as $\boldsymbol{V}_t$. Multiplying $\boldsymbol{F}_w$ to both sides of the equation above,

$$\Delta \boldsymbol{V}_t \boldsymbol{F}_w = -\Delta \log \boldsymbol{p}_{t,K} F_w + \beta \widetilde{\boldsymbol{S}}_K \boldsymbol{\Delta V}_{t+1} \boldsymbol{F}_w \boldsymbol{F}_w.$$

Replacing $\Delta \boldsymbol{V}_t \boldsymbol{F}_w$ and $\Delta \boldsymbol{V}_{t+1} \boldsymbol{F}_w$ by (A.14)

$$\beta^{-1} (\boldsymbol{S}_i - \boldsymbol{S}_K)^{-1} \Delta \boldsymbol{\xi}_{t-1,i,K} = -\Delta \log \boldsymbol{p}_{t,K} \boldsymbol{F}_w + \widetilde{\boldsymbol{S}}_K (\boldsymbol{S}_i - \boldsymbol{S}_K)^{-1} \Delta \boldsymbol{\xi}_{t,i,K} \boldsymbol{F}_w.$$

Equivalently, we have

$$\left[\widetilde{\boldsymbol{S}}_K [\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}, \quad -\beta^{-1} [\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1} \right] \begin{bmatrix} \Delta \boldsymbol{\xi}_{t,i,K} \boldsymbol{F}_w \\ \Delta \boldsymbol{\xi}_{t-1,i,K} \end{bmatrix} = \Delta \log \boldsymbol{p}_{t,K} \boldsymbol{F}_w,$$

where matrix $\begin{bmatrix} \Delta \boldsymbol{\xi}_{t,i,K} \boldsymbol{F}_w \\ \Delta \boldsymbol{\xi}_{t-1,i,K} \end{bmatrix}$ is a $(2J-2)$ by M matrix.

A.5 Proof of Theorem 4

Proof of Lemma 1 Note that the *ex-ante* value function can be represented as

$$V(x,w) = -\log p_K(x,w) + \beta \sum_{x',w'} V(x',w') f(w'|w) s(x'|x,K).$$

With a bit abuse of notation, we have the following matrix representation,

$$V = -\log p_K + \beta \hat{S}_K V \boldsymbol{F}_w$$

where $V \equiv \{V(x=i, w=j)\}_{i,j}$ is a $J \times M$ matrix, p_K is defined analogously. $\hat{S}_K \equiv \{s(x'=j|x=i,K)\}_{i,j}$ is a $J \times J$ matrix. To derive a closed-form expression for the *ex-ante* value function, we first vectorize the matrix expression in the following.

$$\begin{aligned} vec(V) &= vec(-\log p_K + \beta \hat{S}_K V \boldsymbol{F}_w) \\ &= vec(-\log p_K) + \beta[\boldsymbol{F}'_w \otimes \hat{S}_K] vec(V) \end{aligned}$$

As a result,

$$vec(V) = \left[I - \beta(\boldsymbol{F}'_w \otimes \hat{S}_K)\right]^{-1} vec(-\log p_K). \tag{A.16}$$

Note that $\left[I - \beta(\boldsymbol{F}'_w \otimes \hat{S}_K)\right]$ is invertible without imposing any restrictions.

A.6 Exogenous transition of the state variable

We consider identification of subjective expectations under an alternative assumption to Assumption 7 on the transition of state variables. In particular, the state evolution satisfies the following conditions.

Assumption A.6.1 *(a) The observed state variables x and w evolve independently, i.e,*

$$f(x', w'|x, w, a) = f(x'|x, a) f(w'|w, a) = f(x'|x) f(w'|w, a),$$

where $f(w'|w, a)$ is a deterministic function.
(b) Agents believe that the state variables x and w evolve independently and have rational expectations on the evolution of w.

$$s(x', w'|x, w, a) = s(x'|x) s(w'|w, a) = s(x'|x) f(w'|w, a). \tag{A.17}$$

Under Assumption A.6.1, we can represent the log ratio of CCPs in period t as

$$\begin{aligned} \xi_{t,i,K}(x, w) &\equiv \left[u_i(x, w) + \beta \sum_{x', w'} V_{t+1}(x', w') f(w'|w, i) s(x'|x)\right] \\ &\quad - \left[u_K(x, w) + \beta \sum_{x', w'} V_{t+1}(x', w') f(w'|w, K) s(x'|x)\right]. \\ &\equiv u_i(x, w) - u_K(x, w) + \beta S(x) \boldsymbol{V}_{t+1}\left[F_i(w) - F_K(w)\right], \end{aligned} \tag{A.18}$$

where $S(x) \equiv [s(x' = 1|x), ..., s(x' = J|x)]$, $\boldsymbol{V}_t$ is a J by $M-1$ matrix with its (k, j) element being $V_t(x = k, w = j) - V_t(x = k, w = M)$, and $F_a(w) \equiv [f(w' = 1|w, a), ..., f(w' = M - 1|w, a)]'$ with $w' = M$ being excluded as the reference state. The first-difference of log ratio of CCPs is

$$\begin{aligned} \Delta\xi_{t,i,K}(x, w) &\equiv \xi_{t,i,K}(x, w) - \xi_{t-1,i,K}(x, w) \\ &\equiv \beta S(x) \Delta \boldsymbol{V}_{t+1}\left[F_i(w) - F_K(w)\right], \end{aligned}$$

where $\Delta \boldsymbol{V}_{t+1} \equiv \boldsymbol{V}_{t+1} - \boldsymbol{V}_t$. The matrix representation of the equation above is

$$\Delta \boldsymbol{\xi}_{t,i,K} \equiv \beta \boldsymbol{S} \Delta \boldsymbol{V}_{t+1} \big[\boldsymbol{F}_i - \boldsymbol{F}_K\big], \tag{A.19}$$

where $\Delta \boldsymbol{\xi}_{t,i,K}$ is a J by $M-1$ matrix with its (k,j)-th element being $\Delta \xi_{t,i,K}(x=k, w=j)$, $\boldsymbol{S} \equiv [S(x=1), S(x=2), \cdots, S(x=J)]'$, and $\boldsymbol{F}_i \equiv [F_i(w=1), ..., F_i(w=M-1)]$. We then represent the value function recursively by backward induction,

$$\boldsymbol{V}_t = -\log \boldsymbol{p}_{t,K} + \boldsymbol{u}_K + \beta \boldsymbol{S} \boldsymbol{V}_{t+1} \widetilde{\boldsymbol{F}}_K,$$

where $\widetilde{\boldsymbol{F}}_K$ is defined analogy to $\widetilde{\boldsymbol{S}}_K$, and the first-difference of value function also has a recursive representation

$$\Delta \boldsymbol{V}_t = -\Delta \log \boldsymbol{p}_{t,K} + \beta \boldsymbol{S} \boldsymbol{\Delta} \boldsymbol{V}_{t+1} \widetilde{\boldsymbol{F}}_K. \tag{A.20}$$

The derivation of the above equation is analogy to that in equation A.15, so we skip the detail here.

To separate the unknown value function from the subjective expectations, we need to impose the following rank conditions, which is similar to that in assumption (4).

Assumption A.6.2 *Both $\boldsymbol{S}$ and $\boldsymbol{F}_i - \boldsymbol{F}_K$ are of full rank.*

Note that $\boldsymbol{F}_i - \boldsymbol{F}_K$ is a known $M-1$ by $M-1$ matrix, so the full rank assumption is empirically testable.

Combining all moment conditions (A.19) with (A.20) to obtain

$$\beta^{-1} \Delta \boldsymbol{\xi}_{t-1,i,K} \big(\boldsymbol{F}_i - \boldsymbol{F}_K\big)^{-1} = \boldsymbol{S} \left[-\Delta \log \boldsymbol{p}_{t,K} + \Delta \boldsymbol{\xi}_{t,i,K} \big(\boldsymbol{F}_i - \boldsymbol{F}_K\big)^{-1} \widetilde{\boldsymbol{F}}_K\right]. \tag{A.21}$$

The equation above holds for $t = 3, 4, \cdots, T$. We stack all the $T-2$ equations and have

$$\boldsymbol{S} \underbrace{\begin{pmatrix} \left[-\Delta \log \boldsymbol{p}_{3,K} + \Delta \boldsymbol{\xi}_{3,i,K} \big[\boldsymbol{F}_i - \boldsymbol{F}_K\big]^{-1} \widetilde{\boldsymbol{F}}_K\right]' \\ \vdots \\ \left[-\Delta \log \boldsymbol{p}_{T,K} + \Delta \boldsymbol{\xi}_{T,i,K} \big[\boldsymbol{F}_i - \boldsymbol{F}_K\big]^{-1} \widetilde{\boldsymbol{F}}_K\right]' \\ [1, 1, \cdots, 1] \end{pmatrix}'}_{\boldsymbol{A}} = \underbrace{\begin{pmatrix} \beta^{-1} \left[\Delta \boldsymbol{\xi}_{3,i,K} \big(\boldsymbol{F}_i - \boldsymbol{F}_K\big)^{-1}\right]' \\ \vdots \\ \beta^{-1} \left[\Delta \boldsymbol{\xi}_{T,i,K} \big(\boldsymbol{F}_i - \boldsymbol{F}_K\big)^{-1}\right]' \\ [1, 1, \cdots, 1] \end{pmatrix}'}_{\boldsymbol{B}}, \tag{A.22}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are both J by $(T-2)(M-1)+1$ matrices, and $[1, 1, \cdots, 1]$ is a $1 \times J$ vector of ones, which is included to use the fact that every row of the expectations $\boldsymbol{S}$ is a total probability so it adds up to be 1.

Assumption A.6.3 *Matrix $\boldsymbol{A}$ has a full row rank.*

Assumption A.6.3 implicitly requires that $(T-2)(M-1)+1 \geq J$ and this imposes restrictions to time period T and the possible values w takes, M. Under this assumption, the right inverse of matrix $\boldsymbol{A}$ exists and we denote it as $\boldsymbol{A}^+$. We apply the right inverse $\boldsymbol{A}^+$ to (A.22) to get a closed-form solution for $\boldsymbol{S}$:

$$\boldsymbol{S} = \boldsymbol{B}\boldsymbol{A}^+ \tag{A.23}$$

Theorem A.1 *Under Assumption 1-3, A.6.1, A.6.2, and A.6.3, the subjective expectations $s(x'|x)$ for $x, x' \in \{1, 2, ..., J\}$ are identified as a closed-form function of the CCPs $p_t(a|x, w)$ and the objective state transition $f_t(w'|w, a)$ for $t = 1, 2, \cdots, T$, where $T \geq \lceil \frac{J-1}{M-1} \rceil + 2$.*

A.7 Proof of Theorem 5

The first-order Markov process $\{a_t, x_t, \tau\}$ indiciates

$$\begin{aligned} \Pr(a_{t+}, x_{t+1}, a_t, x_t, a_{t-}) &= \sum\nolimits_{\tau=1}^{H} \Pr(a_{t+}|x_{t+1}, \tau) \Pr(x_{t+1}, a_t|x_t, \tau) \Pr(\tau, x_t, a_{t-}), \\ &= \sum\nolimits_{\tau=1}^{H} \Pr(a_{t+}|x_{t+1}, \tau) \Pr(a_t|x_{t+1}, x_t, \tau) \Pr(x_{t+1}|x_t, \tau) \Pr(\tau, x_t, a_{t-}), \end{aligned} \tag{A.24}$$

where $a_{t+} = h(a_{t+l}, ..., a_{t+1})$ and $a_{t-} = h(a_{t-1}, ..., a_{t-l})$.

Note that we have reduced the support of $a_{t+l}, ..., a_{t+1}$ to that of τ by the mapping $h(\cdot)$. We define the following matrices for given x_t, x_{t+1} and $a_t = k$,

$$\begin{aligned} M_{a_{t+},x_{t+1},k,x_t,a_{t-}} &= \Big[\Pr(a_{t+} = i, x_{t+1}, k, x_t, a_{t-} = j)\Big]_{i,j} \\ M_{a_{t+},x_{t+1},\tau} &= \Big[\Pr(a_{t+} = i|x_{t+1}, \tau = j)\Big]_{i,j} \\ M_{\tau,x_t,a_{t-}} &= \Big[\Pr(\tau = i, x_t, a_{t-} = j)\Big]_{i,j} \\ D_{x_{t+1},k|x_t,\tau} &= \text{diag}\Big\{\Pr(x_{t+1}, k|x_t, \tau = 1), ..., \Pr(x_{t+1}, k|x_t, \tau = L)\Big\} \\ D_{k|x_{t+1},x_t,\tau} &= \text{diag}\Big\{\Pr(k|x_{t+1}, x_t, \tau = 1), ..., \Pr(k|x_{t+1}, x_t, \tau = L)\Big\}. \end{aligned}$$

Equation (A.24) can be rewritten as the following matrix form,

$$M_{a_{t+},x_{t+1},k,x_t,a_{t-}} = M_{a_{t+}|x_{t+1},\tau} D_{x_{t+1},k|x_t,\tau} M_{\tau,x_t,a_{t-}}. \tag{A.25}$$

Similarly, we have

$$M_{a_{t+},x_{t+1},x_t,a_{t-}} = M_{a_{t+}|x_{t+1},\tau} D_{x_{t+1}|x_t,\tau} M_{\tau,x_t,a_{t-}}, \tag{A.26}$$

where the matrices are defined analogously to those in (A.25) based on the following equality

$$
\begin{aligned}
& \sum\nolimits_{a_t=1}^{K} \Pr(a_{t+}, x_{t+1}, a_t, x_t, a_{t-}) \\
= & \sum\nolimits_{\tau=1}^{H} \Pr(a_{t+}|x_{t+1}, \tau) \Big[\sum\nolimits_{a_t=1}^{K} \Pr(x_{t+1}, a_t|x_t, \tau) \Big] \Pr(\tau, x_t, a_{t-}) \\
= & \sum\nolimits_{\tau=1}^{H} \Pr(a_{t+}|x_{t+1}, \tau) \Pr(x_{t+1}|x_t, \tau) \Pr(\tau, x_t, a_{t-}),
\end{aligned}
$$

where we use the first-order Markov property of $\{x_t, a_t\}$ to simply $\Pr(x_{t+1}|x_t, \tau, a_{t-})$ as $\Pr(x_{t+1}|x_t, \tau)$.

Consider that

$$
\begin{aligned}
\Pr(a_{t+}, x_{t+1}, x_t, a_{t-}) &= \sum\nolimits_{\tau=1}^{H} \Pr(a_{t+}, x_{t+1}, x_t, a_{t-}, \tau) \\
&= \sum\nolimits_{\tau=1}^{H} \Pr(a_{t+}|x_{t+1}, \tau) \Pr(x_{t+1}|x_t, \tau) \Pr(\tau, x_t, a_{t-}).
\end{aligned}
$$

In matrix representation, it can be rewritten as

$$
M_{a_{t+}, x_{t+1}, x_t, a_{t-}} = M_{a_{t+}|x_{t+1}, \tau} D_{x_{t+1}|x_t, \tau} M_{\tau, x_t, a_{t-}},
$$

where $D_{x_{t+1}|x_t,\tau}$ is a diagonal matrix with its j-th element being $\Pr(x_{t+1}|x_t, \tau = j)$.

Under Assumption 10, the matrix $M_{a_{t+}, x_{t+1}, x_t, a_{t-}}$ for any given x_{t+1} and x_t is of full rank. The equation above implies that $M_{a_{t+}|x_{t+1},\tau}$, $D_{x_{t+1}|x_t,\tau}$ and $M_{\tau,x_t,a_{t-}}$ are all invertible. We take inverse of (A.26) and multiply it from right to (A.25)

$$
\begin{aligned}
M_{a_{t+}, x_{t+1}, k, x_t, a_{t-}} M^{-1}_{a_{t+}, x_{t+1}, x_t, a_{t-}} &= M_{a_{t+}|x_{t+1}, \tau} D_{x_{t+1}, k|x_t, \tau} D^{-1}_{x_{t+1}|x_t, \tau} M^{-1}_{a_{t+}, x_{t+1}, \tau} \\
&= M_{a_{t+}|x_{t+1}, \tau} D_{k|x_{t+1}, x_t, \tau} M^{-1}_{a_{t+}, x_{t+1}, \tau}, \qquad \text{(A.27)}
\end{aligned}
$$

The equation above shows an eigenvalue-eigenvector decomposition of an observed matrix on the left-hand side. Assumptions 11 and 12 guarantee that this decomposition is unique. Therefore, the eigenvector matrix $M_{a_{t+}|x_{t+1},\tau}$, i.e., $\Pr(a_{t+}|x_{t+1}, \tau)$ is identified. We can recover the matrix $M_{\tau,x_t,a_{t-}}$ from (A.26). The distribution $f(x_{t+1}, a_t|x_t, \tau)$, and therefore $\Pr(a_t|x_t, \tau) = p_t(a_t|x_t, \tau)$ by integrate out x_{t+1}, can then identified from equation (A.24) due to the invertibility of matrix $M_{a_{t+}, x_{t+1}, \tau}$.

Table 1: Simulation Results for a DGP of RE: finite horizon

		Estimates with SB				Estimates with rational expectations			
	True	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$
u_1	-2	-2.01	-2.01	-2.00	-2.00	-2.00	-2.00	-2.00	-2.00
		(0.25)	(0.19)	(0.14)	(0.09)	(0.14)	(0.10)	(0.08)	(0.05)
u_2	0.4	0.39	0.40	0.39	0.40	0.40	0.40	0.40	0.40
		(0.23)	(0.16)	(0.13)	(0.08)	(0.15)	(0.10)	(0.08)	(0.05)
u_3	2.1	2.15	2.14	2.12	2.11	2.10	2.10	2.10	2.10
		(0.37)	(0.26)	(0.19)	(0.12)	(0.20)	(0.14)	(0.11)	(0.07)
$\boldsymbol{S}_0(1\|1)$	0.8	0.74	0.74	0.76	0.78				
		(0.26)	(0.24)	(0.20)	(0.15)				
$\boldsymbol{S}_0(2\|1)$	0.1	0.16	0.17	0.17	0.14				
		(0.21)	(0.20)	(0.18)	(0.13)				
$\boldsymbol{S}_0(3\|1)$	0.1	0.10	0.09	0.08	0.08				
		(0.14)	(0.12)	(0.09)	(0.07)				
$\boldsymbol{S}_0(1\|2)$	0.2	0.22	0.21	0.21	0.21				
		(0.31)	(0.29)	(0.27)	(0.23)				
$\boldsymbol{S}_0(2\|2)$	0.6	0.43	0.45	0.46	0.50				
		(0.35)	(0.35)	(0.33)	(0.32)				
$\boldsymbol{S}_0(3\|2)$	0.2	0.35	0.35	0.33	0.29				
		(0.32)	(0.31)	(0.28)	(0.23)				
$\boldsymbol{S}_0(1\|3)$	0.1	0.10	0.09	0.09	0.09				
		(0.12)	(0.10)	(0.10)	(0.08)				
$\boldsymbol{S}_0(2\|3)$	0.19	0.21	0.20	0.20	0.20				
		(0.28)	(0.23)	(0.21)	(0.15)				
$\boldsymbol{S}_0(3\|3)$	0.71	0.69	0.71	0.71	0.71				
		(0.24)	(0.19)	(0.16)	(0.10)				
$\boldsymbol{S}_1(1\|1)$	0.2	0.32	0.31	0.29	0.30				
		(0.31)	(0.29)	(0.27)	(0.24)				
$\boldsymbol{S}_1(2\|1)$	0.6	0.39	0.41	0.46	0.47				
		(0.38)	(0.37)	(0.36)	(0.34)				
$\boldsymbol{S}_1(3\|1)$	0.2	0.30	0.28	0.25	0.24				
		(0.29)	(0.24)	(0.20)	(0.16)				
$\boldsymbol{S}_1(1\|2)$	0.5	0.39	0.39	0.40	0.41				
		(0.36)	(0.32)	(0.30)	(0.24)				
$\boldsymbol{S}_1(2\|2)$	0.2	0.24	0.24	0.25	0.27				
		(0.32)	(0.30)	(0.30)	(0.27)				
$\boldsymbol{S}_1(3\|2)$	0.3	0.37	0.37	0.35	0.33				
		(0.34)	(0.31)	(0.28)	(0.21)				

Note: GDP is rational expectation: $\boldsymbol{S}_0 = \boldsymbol{T}_0$ and $\boldsymbol{S}_1 = \boldsymbol{T}_1$.
Estimation allowing for subjective expectation is with normalization of $\boldsymbol{S}_1(x_3)$, i.e., $\boldsymbol{S}_1(x_3) = \boldsymbol{T}_1(x_3)$.

Table 2: Simulation Results for a DGP of SB: finite horizon

		Estimates with subjective expectations				Estimates with rational expectations			
	True	$N=300$	$N=600$	$N=1000$	$N=2500$	$N=300$	$N=600$	$N=1000$	$N=2500$
u_1	-2	-2.03	-2.02	-2.00	-2.00	-2.29	-2.28	-2.28	-2.28
		(0.24)	(0.17)	(0.13)	(0.09)	(0.14)	(0.10)	(0.08)	(0.05)
u_2	0.4	0.40	0.40	0.39	0.40	0.47	0.47	0.47	0.47
		(0.24)	(0.17)	(0.13)	(0.08)	(0.14)	(0.10)	(0.08)	(0.05)
u_3	2.1	2.13	2.13	2.12	2.11	1.66	1.66	1.66	1.66
		(0.34)	(0.24)	(0.19)	(0.12)	(0.16)	(0.12)	(0.09)	(0.06)
$\boldsymbol{S}_0(1\|1)$	0.9	0.86	0.86	0.88	0.90				
		(0.20)	(0.19)	(0.16)	(0.12)				
$\boldsymbol{S}_0(2\|1)$	0.05	0.08	0.09	0.07	0.05				
		(0.16)	(0.15)	(0.13)	(0.08)				
$\boldsymbol{S}_0(3\|1)$	0.05	0.06	0.05	0.05	0.05				
		(0.10)	(0.09)	(0.07)	(0.05)				
$\boldsymbol{S}_0(1\|2)$	0.1	0.16	0.14	0.15	0.13				
		(0.25)	(0.23)	(0.23)	(0.19)				
$\boldsymbol{S}_0(2\|2)$	0.8	0.57	0.60	0.62	0.65				
		(0.36)	(0.34)	(0.33)	(0.31)				
$\boldsymbol{S}_0(3\|2)$	0.1	0.27	0.27	0.24	0.22				
		(0.29)	(0.27)	(0.23)	(0.22)				
$\boldsymbol{S}_0(1\|3)$	0.05	0.04	0.04	0.04	0.04				
		(0.06)	(0.05)	(0.04)	(0.03)				
$\boldsymbol{S}_0(2\|3)$	0.095	0.12	0.10	0.10	0.10				
		(0.19)	(0.14)	(0.11)	(0.09)				
$\boldsymbol{S}_0(3\|3)$	0.855	0.84	0.86	0.86	0.86				
		(0.19)	(0.13)	(0.10)	(0.08)				
$\boldsymbol{S}_1(1\|1)$	0.6	0.55	0.56	0.57	0.62				
		(0.32)	(0.31)	(0.28)	(0.22)				
$\boldsymbol{S}_1(2\|1)$	0.3	0.32	0.34	0.34	0.28				
		(0.33)	(0.32)	(0.31)	(0.25)				
$\boldsymbol{S}_1(3\|1)$	0.1	0.12	0.11	0.09	0.10				
		(0.18)	(0.14)	(0.11)	(0.09)				
$\boldsymbol{S}_1(1\|2)$	0.25	0.29	0.27	0.28	0.28				
		(0.29)	(0.27)	(0.25)	(0.22)				
$\boldsymbol{S}_1(2\|2)$	0.6	0.41	0.42	0.45	0.45				
		(0.37)	(0.36)	(0.35)	(0.33)				
$\boldsymbol{S}_1(3\|2)$	0.15	0.31	0.31	0.27	0.27				
		(0.29)	(0.27)	(0.25)	(0.21)				

Note: GDP is subjective expectation: $\boldsymbol{S}_0 \neq \boldsymbol{T}_0$ and $\boldsymbol{S}_1 \neq \boldsymbol{T}_1$, but $\boldsymbol{S}_1(x_3) = \boldsymbol{T}_1(x_3)$.
Estimation allowing for subjective expectation is with normalization of $\boldsymbol{S}_1(x_3)$, i.e., $\boldsymbol{S}_1(x_3) = \boldsymbol{T}_1(x_3)$.

Table 3: Simulation Results for a DGP of RE: finite horizon

		Estimates with subjective expectations				Estimates with rational expectations			
	True	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$
u_1	-2	-2.02	-2.02	-2.01	-2.01	-2.00	-2.00	-2.00	-2.00
		(0.21)	(0.15)	(0.11)	(0.07)	(0.14)	(0.10)	(0.08)	(0.05)
u_2	0.4	0.39	0.40	0.39	0.40	0.40	0.40	0.40	0.40
		(0.21)	(0.15)	(0.11)	(0.07)	(0.15)	(0.10)	(0.08)	(0.05)
u_3	2.1	2.14	2.13	2.12	2.11	2.10	2.10	2.10	2.10
		(0.36)	(0.24)	(0.18)	(0.12)	(0.20)	(0.14)	(0.11)	(0.07)
$\boldsymbol{S}_0(1\|1)$	0.8	0.76	0.76	0.76	0.77				
		(0.15)	(0.12)	(0.11)	(0.09)				
$\boldsymbol{S}_0(2\|1)$	0.1	0.17	0.16	0.16	0.15				
		(0.19)	(0.17)	(0.16)	(0.15)				
$\boldsymbol{S}_0(3\|1)$	0.1	0.07	0.07	0.07	0.08				
		(0.08)	(0.07)	(0.07)	(0.06)				
$\boldsymbol{S}_0(1\|2)$	0.2	0.27	0.25	0.26	0.24				
		(0.26)	(0.26)	(0.25)	(0.25)				
$\boldsymbol{S}_0(2\|2)$	0.6	0.48	0.51	0.50	0.52				
		(0.44)	(0.44)	(0.44)	(0.43)				
$\boldsymbol{S}_0(3\|2)$	0.2	0.25	0.24	0.24	0.23				
		(0.20)	(0.19)	(0.19)	(0.18)				
$\boldsymbol{S}_0(1\|3)$	0.1	0.11	0.11	0.10	0.10				
		(0.12)	(0.11)	(0.11)	(0.10)				
$\boldsymbol{S}_0(2\|3)$	0.19	0.20	0.18	0.19	0.19				
		(0.23)	(0.20)	(0.19)	(0.18)				
$\boldsymbol{S}_0(3\|3)$	0.71	0.70	0.71	0.71	0.71				
		(0.17)	(0.13)	(0.10)	(0.09)				

Note: GDP is rational expectation: $\boldsymbol{S}_0 = \boldsymbol{T}_0$ and $\boldsymbol{S}_1 = \boldsymbol{T}_1$.
Estimation allowing for subjective expectation is with normalization of $\boldsymbol{S}_1$, i.e., $\boldsymbol{S}_1 = \boldsymbol{T}_1$.

Table 4: Simulation Results for a DGP of SB: finite horizon

		Estimates with subjective expectations				Estimates with rational expectations			
	True	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$
u_1	-2	-2.01	-2.01	-2.00	-2.01	-1.65	-1.65	-1.64	-1.64
		(0.19)	(0.14)	(0.11)	(0.07)	(0.13)	(0.10)	(0.07)	(0.04)
u_2	0.4	0.40	0.41	0.40	0.41	0.45	0.45	0.44	0.45
		(0.20)	(0.15)	(0.11)	(0.07)	(0.14)	(0.10)	(0.07)	(0.05)
u_3	2.1	2.14	2.13	2.12	2.11	1.72	1.72	1.72	1.72
		(0.33)	(0.23)	(0.18)	(0.11)	(0.17)	(0.12)	(0.09)	(0.06)
$\boldsymbol{S}_0(1\|1)$	0.9	0.88	0.88	0.89	0.89				
		(0.10)	(0.08)	(0.06)	(0.05)				
$\boldsymbol{S}_0(2\|1)$	0.05	0.09	0.08	0.08	0.07				
		(0.12)	(0.10)	(0.09)	(0.08)				
$\boldsymbol{S}_0(3\|1)$	0.05	0.04	0.04	0.04	0.04				
		(0.05)	(0.04)	(0.04)	(0.03)				
$\boldsymbol{S}_0(1\|2)$	0.1	0.25	0.23	0.22	0.19				
		(0.25)	(0.25)	(0.24)	(0.22)				
$\boldsymbol{S}_0(2\|2)$	0.8	0.53	0.58	0.59	0.63				
		(0.43)	(0.42)	(0.42)	(0.39)				
$\boldsymbol{S}_0(3\|2)$	0.1	0.22	0.20	0.19	0.17				
		(0.19)	(0.19)	(0.19)	(0.17)				
$\boldsymbol{S}_0(1\|3)$	0.05	0.05	0.05	0.05	0.05				
		(0.07)	(0.06)	(0.06)	(0.05)				
$\boldsymbol{S}_0(2\|3)$	0.095	0.09	0.09	0.09	0.09				
		(0.13)	(0.11)	(0.10)	(0.09)				
$\boldsymbol{S}_0(3\|3)$	0.855	0.85	0.86	0.85	0.86				
		(0.11)	(0.08)	(0.06)	(0.05)				

Note: GDP is subjective expectation: $\boldsymbol{S}_0 \neq \boldsymbol{T}_0$ but $\boldsymbol{S}_1 = \boldsymbol{T}_1$.
Estimation allowing for subjective expectation is with normalization of $\boldsymbol{S}_1$, i.e., $\boldsymbol{S}_1 = \boldsymbol{T}_1$.

Table 5: Simulation Results for a DGP of RE: infinite horizon

		Estimates with subjective expectations				Estimates with rational expectations			
	True	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$
$u_1(x_1)$	0.1	0.08	0.09	0.09	0.09	0.10	0.11	0.10	0.10
		(0.16)	(0.11)	(0.08)	(0.06)	(0.17)	(0.12)	(0.09)	(0.06)
$u_1(x_2)$	0.2	0.21	0.21	0.21	0.20	0.20	0.20	0.20	0.20
		(0.18)	(0.12)	(0.10)	(0.07)	(0.18)	(0.13)	(0.10)	(0.06)
$u_2(x_1)$	0.2	0.16	0.16	0.17	0.18	0.20	0.20	0.20	0.20
		(0.09)	(0.08)	(0.07)	(0.06)	(0.06)	(0.05)	(0.03)	(0.02)
$u_2(x_2)$	-0.2	-0.18	-0.17	-0.19	-0.20	-0.20	-0.20	-0.20	-0.20
		(0.10)	(0.08)	(0.08)	(0.07)	(0.07)	(0.05)	(0.04)	(0.02)
$\boldsymbol{S}_0(1\|1)$	0.6	0.84	0.81	0.77	0.71				
		(0.33)	(0.33)	(0.36)	(0.36)				
$\boldsymbol{S}_0(2\|1)$	0.45	0.30	0.29	0.37	0.44				
		(0.44)	(0.42)	(0.44)	(0.43)				

Note: GDP is rational expectation: $\boldsymbol{S}_0^x = \boldsymbol{T}_0^x$ and $\boldsymbol{S}_1^x = \boldsymbol{T}_1^x$.
Estimation allowing for subjective expectation is with normalization of $\boldsymbol{S}_1^x$, i.e., $\boldsymbol{S}_1^x = \boldsymbol{T}_1^x$.

Table 6: Simulation Results for a DGP of SB: infinite horizon

		Estimates with subjective expectations				Estimates with rational expectations			
	True	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$
$u_1(x_1)$	0.1	0.09	0.09	0.09	0.10	0.11	0.11	0.11	0.11
		(0.16)	(0.11)	(0.09)	(0.06)	(0.17)	(0.12)	(0.09)	(0.06)
$u_1(x_2)$	0.2	0.21	0.19	0.19	0.19	0.19	0.18	0.19	0.19
		(0.17)	(0.13)	(0.10)	(0.07)	(0.17)	(0.13)	(0.10)	(0.06)
$u_2(x_1)$	0.2	0.18	0.19	0.19	0.19	0.22	0.22	0.22	0.22
		(0.09)	(0.08)	(0.07)	(0.06)	(0.06)	(0.04)	(0.03)	(0.02)
$u_2(x_2)$	-0.2	-0.20	-0.20	-0.21	-0.21	-0.23	-0.22	-0.23	-0.23
		(0.10)	(0.09)	(0.09)	(0.08)	(0.06)	(0.05)	(0.04)	(0.02)
$\boldsymbol{S}_0(1\|1)$	0.7	0.81	0.77	0.76	0.72				
		(0.33)	(0.35)	(0.34)	(0.31)				
$\boldsymbol{S}_0(2\|1)$	0.3	0.28	0.33	0.36	0.35				
		(0.42)	(0.43)	(0.43)	(0.40)				

Note: GDP is subjective expectation: $\boldsymbol{S}_0^x \neq \boldsymbol{T}_0^x$ but $\boldsymbol{S}_1^x = \boldsymbol{T}_1^x$.
Estimation allowing for subjective expectation is with normalization of $\boldsymbol{S}_1^x$, i.e., $\boldsymbol{S}_1^x = \boldsymbol{T}_1^x$.

Table 7: Distribution of observations (by number of years)

# years	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
# obs	101	109	135	105	102	104	102	94	89	94	79	91	85	85	80	71	147
cum %	6.04	12.55	20.6	26.9	33.0	39.2	45.3	50.9	56.3	61.9	66.6	72.0	77.1	82.2	87.0	91.2	100

Table 8: Descriptive Statistics

	# of observations	Mean	Std. Dev.	5-th pctile	median	95-th pctile
age	22, 941	48.84	5.96	40	49	59
education[†]	22, 941	3.98	1.84	0	4	7
annual income (10K $)	22, 941	5.77	6.02	0.67	4.81	13.31
employment	22, 941	.58	.49	0	1	1

Note: The number of observations is aggregated at individual-by-year level. Income is in 1999 dollars. Education is classified into nine groups. 1: 0-5 grades; 2: 6-8 grades; 3: some high school; 4: completed high school; 5: 12 grades plus non-academic training; 6: college, no degree; 7: college, bachelors degree; 8: college, advanced or professional degree, some graduate work; 9: not reported.

Table 9: Estimates of Subjective, Objective Expectations and Preference Parameters

		transition ($a=0$)			transition ($a=1$)			preference	
		low	medium	high	low	medium	high	stationary	ending
	low	1.000	0.000	0.000	0.748	0.249	0.003	-0.243	-0.748
sub.		(0.000)	(0.000)	(0.000)	(0.169)	(0.181)	(0.075)	(0.086)	(0.146)
	medium	0.150	0.850	0.000	0.000	1.000	0.000	0.413	0.149
		(0.058)	(0.186)	(0.156)	(0.000)	(0.000)	(0.000)	(0.041)	(0.099)
	high	0.000	0.999	0.001	—	—	—	0.496	0.272
		(0.049)	(0.186)	(0.168)	—	—	—	(0.200)	(0.472)
	low	0.749	0.249	0.002	0.754	0.244	0.001	-0.611	-0.686
obj.		(0.015)	(0.015)	(0.001)	(0.017)	(0.017)	(0.001)	(0.238)	(0.160)
	medium	0.069	0.921	0.011	0.039	0.947	0.015	0.224	0.111
		(0.004)	(0.004)	(0.002)	(0.002)	(0.003)	(0.002)	(0.049)	(0.112)
	high	0.007	0.237	0.756	0.002	0.294	0.704	0.335	0.132
		(0.005)	(0.037)	(0.037)	(0.002)	(0.032)	(0.033)	(0.169)	(0.580)

Note: The columns “ending” and “stationary” are corresponding to the estimates of flow utility in the terminal period and other periods, respectively.

Table 10: Simulated Conditional Choice Probabilities

	sub. exp.			rational exp.			percentage diff.		
	$x=1$	$x=2$	$x=3$	$x=1$	$x=2$	$x=3$	$x=1$	$x=2$	$x=3$
$t=55$	0.423	0.599	0.614	0.351	0.563	0.581	-17.1%	-6.1%	-5.3%
$t=56$	0.414	0.594	0.610	0.351	0.562	0.581	-15.2%	-5.3%	-4.6%
$t=57$	0.402	0.587	0.604	0.351	0.561	0.582	-12.8%	-4.4%	-3.7%
$t=58$	0.388	0.579	0.597	0.351	0.560	0.582	-9.6%	-3.2%	-2.4%
$t=59$	0.371	0.568	0.584	0.351	0.558	0.583	-5.2%	-1.7%	-0.2%
$t=60$	0.335	0.528	0.533	0.335	0.528	0.533	0.0%	0.0%	0.0%

Note: The percentage difference is defined as [CCPs (sub. exp.)-CCPs (rational. exp.)]/CCPs (sub. exp.).